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IS GAUGE-INVARIANT COMPLETE DECOMPOSITION OF THE NUCLEON SPIN POSSIBLE ?

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Is gauge-invariant complete decomposition of the nucleon spin possible? Although it is a difficult theoretical question which has not reached a complete consensus yet, a general agreement now is that there are at least two physically inequivalent gauge-invariant decompositions (I) and (II) of the nucleon. In these two decompositions, the intrinsic spin parts of quarks and gluons are just common. What discriminate these two decompositions are the orbital angular momentum parts. The orbital angular momenta of quarks and gluons appearing in the decomposition (I) are the so-called “mechanical” orbital angular momenta, while those appearing in the decomposition (II) are the generalized (gauge-invariant) “canonical” ones. By this reason, these decompositions are also called the “mechanical” and “canonical” decompositions of the nucleon spin, respectively. A crucially important question is which decomposition is more favorable from the observational viewpoint. The main objective of this concise review is to try to answer this question with careful consideration of recent intensive researches on this problem.

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1. Introduction

The so-called “nucleon spin puzzle” raised by the epoch-making measurements by the EMC Collaboration in 1987 is still one of the most fundamental problems in quantum chromodynamics (QCD).^{1,2} (General review of the nucleon spin problem, can, for example, be found in Refs. 3, 4, 5, 6, 7, 8, 9, 10.) In the past ten years, there have been several remarkable progresses on this problem from the experimental side. First, a lot of experimental evidence has been accumulated, which indicates that the gluon polarization inside the nucleon would not be extremely large.^{11–14} At the least, now it seems widely accepted that the $U_A(1)$ -anomaly motivated explanation of the nucleon spin puzzle^{15–17} is disfavored. Second, the quark spin fraction or the net longitudinal quark polarization Δq has been fairly precisely determined through high-statistics measurements of the deuteron spin structure function by COMPASS^{18,19} and the HERMES group.²⁰ According to these analyses, the portion of the nucleon spin coming from the intrinsic quark spin turned out to be around 1/3. What carries the rest 2/3 of the nucleon spin, then ? This is a fundamental question of QCD, which we must answer. To answer this question unambiguously, we cannot avoid to clarify the following issues :

- What is a precise (QCD) definition of each term of the decomposition ?
- How can we extract individual term by means of direct measurements ?

Since QCD is a color SU(3) gauge theory, and because the general principle of physics dictates that gauge-invariance is a necessary condition of observability, the gauge-invariance is believed to play a crucially important role in the nucleon spin decomposition problem.^{21–27} In the past several years, this interesting but difficult problem has been an object of intense debate.^{28–81} Very recently, Leader and Lorcé wrote a fairly extensive first review, which nicely summarizes the current status of the nucleon spin decomposition problem.⁸² The purpose of the present shorter review is to give a general survey on the same problem from somewhat different viewpoint from theirs.

The broad guideline of the paper is as follows. First, in Sect.2, we start with concisely reviewing the history of the nucleon spin decomposition problem. It is explained why there exist two totally different decompositions of the nucleon spin and how they are different. Next, in Sect.2, we shall dwell on controversial theoretical issues on the gauge-invariant decomposition problem of the nucleon spin. We point out that there are conceptually opposing two approaches to the problem. The one is the standard gauge-fixing approach, while the other is the so-called gauge-invariant-extension (GIE) approach. A critical difference between these two ways of thinking is explained in detail. In Sect. 4, we briefly introduce the recent controversies on the transverse nucleon spin decomposition. The main focus there is put on clarifying a significant difference between the transverse spin sum rule and the longitudinal spin sum rule. In Sect.5, we shall look at the question whether the canonical orbital angular momentum (OAM) truly satisfies the standard SU(2) commutation rela-

tion. Contrary to wide-spread belief, it is shown that neither the canonical OAM nor the intrinsic spin of the massless gauge field satisfies the SU(2) commutation relation, and that this fact is inseparably connected with the vanishing mass of the photon or the gluon. An important conclusion drawn from this consideration is that only the longitudinal component of the total gluon angular momentum can be decomposed into its intrinsic spin and orbital OAM parts in a gauge- and frame-independent way. What characterizes the difference between the two different OAMs of the quarks and gluons is the potential angular momentum in the terminology of Ref. 41. To understand its physical meaning is of vital importance for answering the question “Which of the canonical OAM or the mechanical OAM is a *physical* quantity from the viewpoint of observability?” Naturally, only the high-energy deep-inelastic-scattering (DIS) measurements, which raised the problem in the first place, would provide us with a possible practical means to answer the proposed question on the nucleon spin decompositions. Section 7 is therefore devoted to the most important issue of the nucleon spin decomposition problem : how can we relate the two existing nucleon spin decompositions, i.e. the mechanical decomposition and the canonical decomposition, to direct high-energy deep-inelastic-scattering (DIS) observables? We try to answer this question with the help of several recent researches in this direction. Finally, in Sect.8, we shall summarize what we have learned and make some concluding remarks.

2. A Brief History of the Nucleon Spin Decomposition Problem

The existence of two different decompositions of the nucleon spin has been long known in the QCD spin physics community. One is the Jaffe-Manohar decomposition given as²¹

$$\begin{aligned} J_{QCD} = & \int \psi^\dagger \frac{1}{2} \boldsymbol{\Sigma} \psi d^3x + \int \psi^\dagger \mathbf{x} \times \frac{1}{i} \nabla \psi d^3x \\ & + \int \mathbf{E}^a \times \mathbf{A}^a d^3x + \int E^{ai} \mathbf{x} \times \nabla A^{ai} d^3x, \end{aligned} \quad (1)$$

with a being the color index of the gluon field, while the other is the Ji decomposition given as follows^{22, 23} :

$$\begin{aligned} J_{QCD} = & \int \psi^\dagger \frac{1}{2} \boldsymbol{\Sigma} \psi d^3x + \int \psi^\dagger \mathbf{x} \times \frac{1}{i} \mathbf{D} \psi d^3x \\ & + \int \mathbf{x} \times (\mathbf{E}^a \times \mathbf{B}^a) d^3x, \end{aligned} \quad (2)$$

where \mathbf{D} is the standard covariant derivative defined by $\mathbf{D} \equiv \nabla - i g \mathbf{A}$. In these popular decompositions, only the intrinsic quark spin part is common, and the other parts are all different. (See Fig.1.) An apparent disadvantage of the Jaffe-Manohar decomposition is that each term is not separately gauge-invariant, except for the quark spin part. On the other hand, each term of the Ji decomposition is separately gauge-invariant. Unfortunately, it was claimed and has been long believed

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that further gauge-invariant decomposition of the total gluon angular momentum into its spin and orbital parts is impossible in this widely-known gauge-invariant decomposition.

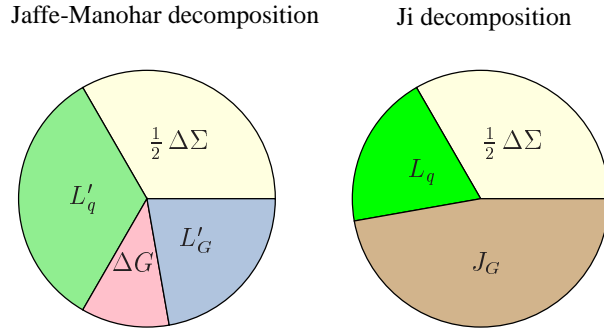


Fig. 1. Two widely-known nucleon spin decompositions.

An especially annoying observation was that, since the quark orbital angular momenta (OAMs) in the two decompositions are apparently different, i.e.

$$L'_q \neq L_q, \quad (3)$$

one must inevitably conclude that the sum of the gluon spin and the gluon OAM in the Jaffe-Manohar decomposition does not coincide with the total gluon angular momentum in the Ji decomposition,

$$\Delta G + L'_G \neq J_G. \quad (4)$$

Intensive debates began several years ago when Chen et al.^{28,29} proposed a new gauge-invariant decomposition of nucleon spin. The basic idea is to decompose the gluon field \mathbf{A} into two parts, i.e. the physical component \mathbf{A}_{phys} and the pure-gauge one \mathbf{A}_{pure} . Under general gauge transformations $U(x)$, the physical part is supposed to transform covariantly,

$$\mathbf{A}_{phys}(x) \rightarrow \mathbf{A}'(x) = U(x) \mathbf{A}_{phys}(x) U^\dagger(x), \quad (5)$$

while the pure-gauge part is required to transform inhomogeneously, i.e. as

$$\mathbf{A}_{pure}(x) \rightarrow \mathbf{A}'_{pure}(x) = U(x) \left(\mathbf{A}_{pure}(x) + \frac{i}{g} \nabla \right) U^\dagger(x). \quad (6)$$

We recall that, in the case of quantum electrodynamics (QED), their decomposition is nothing but the familiar transverse-longitudinal decomposition of the vector potential $\mathbf{A}(x)$ given as

$$\mathbf{A}(x) = \mathbf{A}_\perp(x) + \mathbf{A}_\parallel(x), \quad (7)$$

with the conditions :

$$\nabla \cdot \mathbf{A}_\perp(x) = 0, \quad \nabla \times \mathbf{A}_\parallel(x) = 0. \quad (8)$$

As is well-known, this decomposition is *unique* owing to the famous Helmholtz theorem, once the Lorentz frame of reference is fixed.^a In the case of quantum chromodynamics (QCD), a similar decomposition is not so simple as the abelian case. To uniquely specify the decomposition, Chen et al. impose an additional condition for the physical component,

$$[\mathbf{A}_{phys}, \mathbf{E}] \equiv \mathbf{A}_{phys} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{A}_{phys} = 0, \quad (9)$$

where \mathbf{E} is the color electric field. (An alternative and simpler condition for fixing the physical component was proposed later by themselves³³ and also by Zhou, Huang and Huang⁷¹ as a non-Abelian generalization of the transversality condition $\nabla \cdot \mathbf{A}_{phys} = 0$ for the Abelian vector potential. It is given by $\mathbf{D}_{pure} \cdot \mathbf{A} = 0$ with $\mathbf{D}_{pure} \equiv \nabla - i g [\mathbf{A}_{pure}, \cdot]$ being the pure-gauge covariant derivative for the adjoint representation.) In any case, after imposing such an additional condition, which is supposed to uniquely fix the decomposition of the gluon field \mathbf{A} into \mathbf{A}_{phys} and \mathbf{A}_{pure} , Chen et al. proposed the following decomposition of the nucleon spin :

$$\mathbf{J}_{QCD} = \mathbf{S}'_q + \mathbf{L}'_q + \mathbf{S}'_G + \mathbf{L}'_G, \quad (10)$$

where

$$\mathbf{S}'_q = \int \psi^\dagger \frac{1}{2} \boldsymbol{\Sigma} \psi d^3x, \quad (11)$$

$$\mathbf{L}'_q = \int \psi^\dagger \mathbf{x} \times \left(\frac{1}{i} \nabla - g \mathbf{A}_{pure} \right) \psi d^3x, \quad (12)$$

$$\mathbf{S}'_G = \int \mathbf{E}^a \times \mathbf{A}_{phys}^a d^3x, \quad (13)$$

$$\mathbf{L}'_G = \int E^{aj} (\mathbf{x} \times \mathbf{D}_{pure}) A_{phys}^{aj} d^3x, \quad (14)$$

with $\mathbf{D}_{pure} = \nabla - i g \mathbf{A}_{pure}$. A remarkable feature of this decomposition is that each term is separately gauge-invariant, as can easily be verified from the above-mentioned covariant and inhomogeneous gauge transformation properties of the physical and pure-gauge components of the gluon. Also noteworthy is that it reduces to the *gauge-variant* decomposition of Jaffe and Manohar in a particular gauge, $\mathbf{A}_{pure} = 0$, and $\mathbf{A} = \mathbf{A}_{phys}$.

Soon after, however, Wakamatsu pointed out that the way of gauge-invariant complete decomposition of nucleon spin is not necessarily unique, and proposed yet another decomposition of the nucleon spin given by⁴¹

$$\mathbf{J}_{QCD} = \mathbf{S}_q + \mathbf{L}_q + \mathbf{S}_G + \mathbf{L}_G, \quad (15)$$

^aTo be more precise, the uniqueness of the decomposition is guaranteed by a supplemental condition that the gauge field \mathbf{A} falls off faster than $1/r^2$ at the spatial infinity. This condition is not necessarily satisfied in some singular gauges like the light-cone gauge.

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where

$$\mathbf{S}_q = \int \psi^\dagger \frac{1}{2} \boldsymbol{\Sigma} \psi d^3x, \quad (16)$$

$$\mathbf{L}_q = \int \psi^\dagger \mathbf{x} \times \left(\frac{1}{i} \nabla - g \mathbf{A} \right) \psi d^3x, \quad (17)$$

$$\mathbf{S}_G = \int \mathbf{E}^a \times \mathbf{A}_{phys}^a d^3x, \quad (18)$$

$$\mathbf{L}_G = \int E^{aj} (\mathbf{x} \times \mathbf{D}_{pure}) A_{phys}^{aj} d^3x + \int \rho^a (\mathbf{x} \times \mathbf{A}_{phys}^a) d^3x. \quad (19)$$

The characteristic features of this decomposition are as follows. First, the quark parts of this decomposition (both of spin and orbital parts) are just common with the Ji decomposition. Second, the quark and gluon intrinsic spin parts are common with the Chen decomposition. A crucial difference with the Chen decomposition resides in the orbital parts. Namely, although the sums of the quark and gluon OAMs in the two decompositions are the same, i.e.

$$\mathbf{L}_q + \mathbf{L}_G = \mathbf{L}'_q + \mathbf{L}'_G, \quad (20)$$

each term is different in such a way that

$$\mathbf{L}_G - \mathbf{L}'_G = -(\mathbf{L}_q - \mathbf{L}'_q) = \int \rho^a (\mathbf{x} \times \mathbf{A}_{phys}^a) d^3x \equiv \mathbf{L}_{pot}. \quad (21)$$

The difference arises from the treatment of the 2nd term of Eq.(19). He call this term the *potential angular momentum* \mathbf{L}_{pot} , because the QED correspondent of this term is the orbital angular momentum (OAM) carried by the electromagnetic field or potential.^b Wakamatsu includes this term in the *gluon* OAM part, while Chen et al. include it in the *quark* OAM part.

To understand the difference more clearly, let us first recall the fact that the potential angular momentum term can also be expressed as

$$\int \rho^a (\mathbf{x} \times \mathbf{A}_{phys}^a) d^3x = g \int \psi^\dagger \mathbf{x} \times \mathbf{A}_{phys} \psi d^3x. \quad (22)$$

Note that this term is solely gauge-invariant, as can easily be convinced from the covariant (or homogeneous) gauge transformation property of the physical part of the gluon field \mathbf{A}_{phys} . This means that the gauge principle alone cannot say in which part of the decomposition, one should include the potential angular momentum term. One certainly has a freedom to include it into the quark OAM part as well, which would lead to the Chen decomposition. In fact, if one adds the potential angular momentum to the quark OAM term of the Ji (or Wakamatsu) decomposition, the physical part of \mathbf{A} is exactly canceled out and the pure gauge part is

^bThis is just the quantity appearing in the Feynman paradox raised in his famous textbook of classical electrodynamics.⁸³

left, which just leads to the quark OAM term of the Chen decomposition in the following manner :

$$\begin{aligned}
& \mathbf{L}_q \text{ (Ji or Wakamatsu)} + \mathbf{L}_{pot} \\
&= \int \psi^\dagger \mathbf{x} \times \left(\frac{1}{i} \nabla - g \mathbf{A} \right) \psi d^3x + g \int \psi^\dagger \mathbf{x} \times \mathbf{A}_{phys} \psi d^3x \\
&= \int \psi^\dagger \mathbf{x} \times \left(\frac{1}{i} \nabla - g \mathbf{A}_{pure} \right) \psi d^3x = \mathbf{L}'_q \text{ (Chen)}. \tag{23}
\end{aligned}$$

It seems true that the two complete decompositions of the nucleon spin, i.e. the one due to Chen et al. and the other due to Wakamatsu, are both gauge-invariant. However, a disadvantage of these decompositions is that they are given in noncovariant forms. This is not convenient, for example, if one tries to connect these decompositions with high-energy DIS observables. Also from more general viewpoint, the non-covariant treatment makes it hard to check out the Lorentz-frame dependence or independence of the nucleon spin sum rule derived on the basis of them. The “seemingly” covariant generalization of the gauge-invariant decomposition was given by Wakamatsu.^{42c} The starting point of this proposal is the formally covariant decomposition of the full gauge field A^μ into its physical and pure-gauge parts, i.e. $A^\mu = A^\mu_{phys} + A^\mu_{pure}$. He showed that the gauge-invariant decomposition of the QCD angular momentum tensor can be obtained with use of the following three conditions only. The first is the pure-gauge condition for A^μ_{pure} ,

$$F^{\mu\nu}_{pure} \equiv \partial^\mu A^\nu_{pure} - \partial^\nu A^\mu_{pure} - i g [A^\mu_{pure}, A^\nu_{pure}] = 0, \tag{24}$$

while the second and the third are the gauge transformation properties for these two components :

$$A^\mu_{phys}(x) \rightarrow U(x) A^\mu_{phys}(x) U^{-1}(x), \tag{25}$$

$$A^\mu_{pure}(x) \rightarrow U(x) \left(A^\mu_{pure}(x) - \frac{i}{g} \partial^\mu \right) U^{-1}(x). \tag{26}$$

As a matter of course, these conditions are not enough to fix the decomposition uniquely. This is not unrelated to the fact that there are many gauge choices, which nevertheless leads to the same answer for gauge-invariant observables.^d The point of his argument was therefore that one can postpone a concrete specification of the decomposition until later stage, while accomplishing a gauge-invariant decomposition of the QCD angular momentum tensor $M^{\mu\nu\lambda}$ based on the above conditions only. As anticipated, he found the existence of two different gauge-invariant decompositions, which he calls the decomposition (I) and the decomposition (II). Let us

^cThe word “seemingly” here means that the decomposition $A^\mu = A^\mu_{phys} + A^\mu_{pure}$, although being covariantly-looking, is intrinsically noncovariant, since it is essentially a transverse-longitudinal decomposition, while what is transverse depends on the choice of Lorentz-frame of reference.

^dRemember that a gauge-fixing procedure amounts to a process of eliminating unphysical gauge degrees of freedom, thereby selecting out physical degrees of freedom of the gauge field.

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start with the decomposition (II). The decomposition (II) is a “seemingly” covariant generalization of the Chen decomposition represented as

$$M_{QCD}^{\mu\nu\lambda} = M_{q-spin}^{\prime\mu\nu\lambda} + M_{q-OAM}^{\prime\mu\nu\lambda} + M_{G-spin}^{\prime\mu\nu\lambda} + M_{G-OAM}^{\prime\mu\nu\lambda} + \text{boost} + \text{total divergence}, \quad (27)$$

with

$$M_{q-spin}^{\prime\mu\nu\lambda} = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} \bar{\psi} \gamma_\sigma \gamma_5 \psi, \quad (28)$$

$$M_{q-OAM}^{\prime\mu\nu\lambda} = \bar{\psi} \gamma^\mu (x^\nu i D_{pure}^\lambda - x^\lambda i D_{pure}^\nu) \psi, \quad (29)$$

$$M_{G-spin}^{\prime\mu\nu\lambda} = 2 \text{Tr} \{ F^{\mu\lambda} A_{phys}^\nu - F^{\mu\nu} A_{phys}^\lambda \}, \quad (30)$$

$$M_{G-OAM}^{\prime\mu\nu\lambda} = 2 \text{Tr} \{ F^{\mu\alpha} (x^\nu D_{pure}^\lambda - x^\lambda D_{pure}^\nu) A_\alpha^{phys} \}. \quad (31)$$

As one sees, the quark and gluon OAMs appearing in this decomposition are the canonical OAMs aside from the unphysical gauge degrees of freedom. By this reason, it is sometimes called the “canonical” decomposition of the nucleon spin.

On the other hand, the decomposition (I) is a “seemingly” covariant generalization of another noncovariant decomposition due to Wakamatsu.⁴¹ It is given as

$$M^{\mu\nu\lambda} = M_{q-spin}^{\mu\nu\lambda} + M_{q-OAM}^{\mu\nu\lambda} + M_{G-spin}^{\mu\nu\lambda} + M_{G-OAM}^{\mu\nu\lambda} + \text{boost} + \text{total divergence}, \quad (32)$$

with

$$M_{q-spin}^{\mu\nu\lambda} = M_{q-spin}^{\prime\mu\nu\lambda}, \quad (33)$$

$$M_{q-OAM}^{\mu\nu\lambda} = \bar{\psi} \gamma^\mu (x^\nu i D^\lambda - x^\lambda i D^\nu) \psi, \quad (34)$$

$$M_{G-spin}^{\mu\nu\lambda} = M_{G-spin}^{\prime\mu\nu\lambda}, \quad (35)$$

$$M_{G-OAM}^{\mu\nu\lambda} = M_{G-OAM}^{\prime\mu\nu\lambda} + 2 \text{Tr} [(D_\alpha F^{\alpha\mu}) (x^\nu A_{phys}^\lambda - x^\lambda A_{phys}^\nu)]. \quad (36)$$

The quark and gluon OAMs appearing in this decomposition is the mechanical OAMs, so that it is reasonable to call it the “mechanical” decomposition of the nucleon spin.

One of the greatest advantages of the “seemingly” covariant generalization is that it generalizes and unifies the various nucleon spin decomposition in the market. For example, it was emphasized that the canonical decomposition (II) reduces to any ones of Bashinsky-Jaffe,²⁵ of Chen et al.,^{28,29} and of Jaffe-Manohar,²¹ after an appropriate gauge-fixing in a suitable Lorentz frame, which appears to indicate that they are essentially the same decompositions.^e Another advantage of the covariant formulation of the nucleon spin decomposition is that, it becomes easier to establish explicit relations with high-energy DIS observables. In fact, as pointed

^eThis statement would not be correct in the most general context, but it is most probably correct when applied to the most important longitudinal nucleon spin decomposition, as we shall discuss throughout this paper.

out in Ref. 42, the Bashinsky-Jaffe decomposition obtained based on the light-cone gauge is a special case of this general treatment. In Ref. 42, this fact was utilized to show that the quark and gluon intrinsic spin parts of the above covariant decomposition precisely coincides with the first moments of the polarized distribution functions appearing in the polarized DIS cross-sections.

$$\Delta q = \int \Delta q(x) dx, \quad \Delta g = \int \Delta g(x) dx. \quad (37)$$

Furthermore, based on the mechanical decomposition (I), it can be shown that the following important relations hold⁴² :

$$\begin{aligned} L_q &= \langle p \uparrow | M_{q-OAM}^{012} | p \uparrow \rangle / \langle p \uparrow | p \uparrow \rangle \\ &= \frac{1}{2} \int x [H^q(x, 0, 0) + E^q(x, 0, 0)] dx - \frac{1}{2} \int \Delta q(x) dx, \end{aligned} \quad (38)$$

with

$$M_{q-OAM}^{012} = \bar{\psi} \left(\mathbf{x} \times \frac{1}{i} \mathbf{D} \right)^3 \psi, \quad (39)$$

and also

$$\begin{aligned} L_G &= \langle p \uparrow | M_{G-OAM}^{012} | p \uparrow \rangle / \langle p \uparrow | p \uparrow \rangle \\ &= \frac{1}{2} \int x [H^G(x, 0, 0) + E^G(x, 0, 0)] dx - \int \Delta g(x) dx, \end{aligned} \quad (40)$$

with

$$M_{G-OAM}^{012} = 2 \text{Tr} [E^j (\mathbf{x} \times \mathbf{D}_{pure})^3 A_j^{phys}] + 2 \text{Tr} [\rho (\mathbf{x} \times \mathbf{A}_{phys})^3]. \quad (41)$$

The relation (38) with (39) means that the quantity defined as the difference between the second moments of unpolarized GPDs $H + E$ and the first moment of polarized quark distribution just coincides with the proton matrix element of the quark OAM operator containing full gauge covariant derivative.²² This just confirms Ji's observation that the quark OAM extracted from the combined analysis of GPDs and polarized PDFs is the *mechanical* OAM not the *canonical* OAM ! The equality (40) with (41) provides us with new information. It tells that the gluon OAM extracted from the combined analysis of GPD and polarized PDF contains the *potential* OAM, in addition to the gluon *canonical* OAM. We have pointed out before that the sum of the gluon intrinsic spin and the gluon OAM in the Jaffe-Manohar decomposition does not coincide with the gluon total angular momentum in the Ji decomposition. (See Eq.(4).) The reason of this observation is self-evident now. It is just the *potential angular momentum* that compensates this discrepancy in such a way that

$$\Delta G + L'_G = J_G - L_{pot}. \quad (42)$$

3. On the Gauge-Invariant-Extension Approach

In the previous section, it was pointed out that the three conditions (24), (25) and (26) are not sufficient to uniquely fix the “seemingly” covariant decomposition of the gauge field, $A^\mu(x) = A_{phys}^\mu(x) + A_{pure}^\mu(x)$. This is only natural, because any physical condition necessary for fixing the physical component of A^μ has not been imposed at this stage. Nevertheless, the viewpoint advocated in⁴² is that, since each term of the decomposition (I) and (II) are clearly gauge-invariant, one can make any desired gauge choice at the later stage as the needs arise. However, quite a different view has rapidly spread out around the community.^{?, 52–55, 74, 75, 82} According to this viewpoint, the Chen decomposition is a gauge-invariant-extension (GIE) of the Jaffe-Manohar decomposition based on the Coulomb gauge, while the Bashinsky-Jaffe decomposition (or the Hatta decomposition) is another GIE of the Jaffe-Manohar decomposition based on the light-cone gauge. The claim is that, since they are different GIEs, there is no reason that they give the same physical predictions.

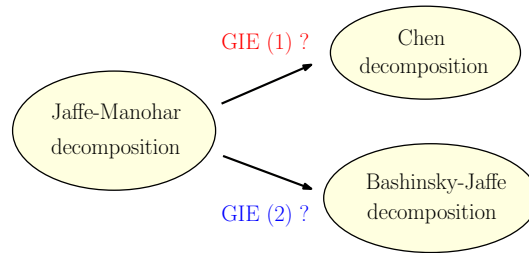


Fig. 2. A schematic picture of the idea of gauge-invariant extension.

However, the strangeness of the idea of GIE seems already clear from the following simple consideration of conceptual nature. Suppose that the Chen decomposition and the Bashinsky-Jaffe decomposition (or the Hatta decomposition) are two physically inequivalent GIEs of the Jaffe-Manohar decomposition. (See Fig.2.) Then, one would immediately encounter the following conceptual questions.

- What are the *physical meaning* of extended gauge symmetries ?
- Are there *plural* color gauge symmetries in nature ?

Since QCD is a theory with color gauge-invariance from the start, it is obvious that there is no need of introducing the idea like GIE. In fact, at least in the simpler case of QED, it was explicitly shown in Ref. 45 that the Chen decomposition is *never* a GIE. It simply utilizes the gauge degrees of freedom which are present from the start in the original theoretical expression of the total angular momentum of a coupled system of the photon and the charged particles.

In the history of the nucleon spin decomposition problem, the word gGIEh was

first introduced by Hoodbhoy and Ji in Ref. 84. It was used to explain that the “should-be” observable gluon spin ΔG can be thought of as a gauge-invariant of the Jaffe-Manohar gluon spin, which is not manifestly gauge-invariant. The idea was revised in more recent paper by Ji, Xu, and Zhao.⁷³ A similar idea of GIE has been pursued further by Lorcé based on the geometrical formulation of gauge theories.^{7,51–55} According to Lorcé, due to the existence of hidden symmetry called the Stückelberg symmetry, there are in principle infinitely many decompositions of the nucleon spin.

Before explaining his idea in more detail, we point out that the word “Stückelberg symmetry” is a little disturbing. The original motivation of Stückelberg’s idea was to show that some non-gauge theory can be made a gauge theory by using the so-called Stückelberg trick.^{85–87} The simplest example is provided by the theory of massive neutral vector boson, the lagrangian of which is given by

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}m^2 A_\mu^2. \quad (43)$$

The mass term in this lagrangian obviously breaks the invariance under the gauge transformation,

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x). \quad (44)$$

The “Stückelberg trick” begins with introducing an auxiliary scalar field $\phi(x)$ called the “compensator” *by hand*,

$$\mathcal{L} \rightarrow \mathcal{L}' = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}m^2 (A_\mu - \partial_\mu \phi)^2. \quad (45)$$

If $\phi(x)$ transforms as follows under the gauge transformation,

$$\phi(x) \rightarrow \phi(x) + \Lambda(x), \quad (46)$$

then, the new lagrangian \mathcal{L}' is clearly gauge-invariant! In this theoretical framework, the original lagrangian \mathcal{L} is thought to be a *gauge-fixed form* ($\phi = 0$) of the new gauge-invariant lagrangian \mathcal{L}' . It is clear that this trick enables us to make wide class of nongauge theories into theories with gauge degrees of freedom, so that it makes the distinction between true gauge theories and the artificially constructed gauge theories obscure.

Now let us come back to Lorcé’s argument. For clarity, we discuss for a while simpler case of abelian gauge theory. According to him, starting from some decomposition of the gauge field $A^\mu(x) = A_{phys}^\mu(x) + A_{pure}^\mu(x)$, one can get another perfectly acceptable decomposition $A^\mu(x) = \bar{A}_{phys}^\mu(x) + \bar{A}_{pure}^\mu(x)$, where

$$\bar{A}_{phys}^\mu(x) = A_{phys}^\mu(x) - \partial^\mu C(x), \quad (47)$$

$$\bar{A}_{pure}^\mu(x) = A_{pure}^\mu(x) + \partial^\mu C(x), \quad (48)$$

with $C(x)$ being an arbitrary function of space and time. This then means that there are infinitely many decompositions of the gauge field into the physical and

pure-gauge components. It is certainly true that this transformation preserves the pure-gauge-condition of the pure-gauge component. i.e.

$$\bar{F}^{\mu\nu}(x) = \partial^\mu \bar{A}_{pure}^\nu(x) - \partial^\nu \bar{A}_{pure}^\mu(x) = 0, \quad (49)$$

and also its transformation property

$$\bar{A}_{pure}^\mu(x) \rightarrow \bar{A}_{pure}^\mu(x) + \partial^\mu \Lambda(x), \quad (50)$$

under general gauge transformation $A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu \Lambda(x)$. However, as emphasized in Refs. 45 and 46, this argument does not pay enough attention to the fact why $A_\mu^{phys}(x)$ is named the physical component. As repeatedly emphasized, in the QED case with the noncovariant treatment, the decomposition by Chen et al. is nothing but the standard decomposition of the vector potential \mathbf{A} into the transverse and longitudinal components as

$$\mathbf{A}(x) = \mathbf{A}_\perp(x) + \mathbf{A}_\parallel(x), \quad (51)$$

where the two components are respectively required to satisfy the divergence-free and irrotational conditions :

$$\nabla \cdot \mathbf{A}_\perp = 0, \quad \nabla \times \mathbf{A}_\parallel = 0. \quad (52)$$

It is easy to verify that these two components transform as

$$\mathbf{A}_\perp(x) \rightarrow \mathbf{A}'_\perp(x) = \mathbf{A}_\perp(x), \quad (53)$$

$$\mathbf{A}_\parallel(x) \rightarrow \mathbf{A}'_\parallel(x) = \mathbf{A}_\parallel(x) - \nabla \Lambda(x), \quad (54)$$

under general gauge transformations. This show that, while \mathbf{A}_\parallel is a totally *arbitrary* quantity that can be changed freely by gauge transformation, \mathbf{A}_\perp is essentially a unique object with definite physical entity. In fact, within the above noncovariant framework, the Stückelberg transformation *a la* Lorcé reduces to^{52,53}

$$\mathbf{A}_\perp(x) \rightarrow \bar{\mathbf{A}}_\perp(x) = \mathbf{A}_\perp(x) + \nabla C(x), \quad (55)$$

$$\mathbf{A}_\parallel(x) \rightarrow \bar{\mathbf{A}}_\parallel(x) = \mathbf{A}_\parallel(x) - \nabla C(x). \quad (56)$$

One can see that the transformed longitudinal component $\bar{\mathbf{A}}_\parallel(x)$ retains the irrotational property,

$$\nabla \times \bar{\mathbf{A}}_\parallel = \nabla \times (\mathbf{A}_\parallel - \nabla C(x)) = \nabla \times \mathbf{A}_\parallel = 0. \quad (57)$$

(This is simply a reflection of the fact that the standard gauge transformation for \mathbf{A}_\parallel keeps the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ intact.) In contrast, one finds that the transformed component $\bar{\mathbf{A}}_\perp$ does not satisfy the desired divergence-free (or *transversality*) condition $\nabla \cdot \bar{\mathbf{A}}_\perp = 0$ any more, since

$$\nabla \cdot \bar{\mathbf{A}}_\perp(x) = \nabla \cdot (\mathbf{A}_\perp(x) + \nabla C(x)) = \Delta C(x) \neq 0, \quad (58)$$

unless $\Delta C(x) = 0$. The condition $\Delta C(x) = 0$ means that $C(x)$ is a harmonic function in three spatial dimension. If it is required to vanish at the spatial infinity, it must be identically zero, i.e. $C(x) \equiv 0$. This means that the invariance under

the Stückelberg transformation actually does not exist, provided that an appropriate physical condition, i.e. the transversality condition, is imposed on the physical component of the vector potential \mathbf{A} . To be more strict, there are some matters to be attended. The boundary condition $\lim_{|x| \rightarrow \infty} C(x) = 0$ may not be always satisfied in some singular gauges like the light-cone gauge. More importantly, the notion of transversality depends on the Lorentz frame of reference. Namely, a vector field that appears transverse may not necessarily be transverse in another Lorentz frame. This indicates that, if the Stückelberg symmetry as proposed by Lorcé *does* exist, one of its origin would be the Lorentz-frame dependence of the decomposition $A^\mu(x) = A_{phys}^\mu(x) + A_{pure}^\mu(x)$.

In a recent paper,⁵⁵ Lorcé argued about another possible origins of the Stückelberg symmetry. One is the path-dependence of the decomposition $A^\mu(x) = A_{phys}^\mu(x) + A_{pure}^\mu(x)$, which arises within the formulation using the gauge link or the Wilson line. (In a recent paper, Tiwari discussed the role of topology in the path-dependence.³⁸) The other is the background dependence of the decomposition, which arises in the formulation based on the background field method of gauge theories. In the following, we discuss only the former, because it is more closely connected with actual physical situations which we encounter in the studies of high-energy DIS observables. By making use of a path-dependent Wilson line $\mathcal{W}_C(x, x_0) = \mathcal{P} \left[e^{i g \int_{x_0}^x A_\mu(s) ds^\mu} \right]$ with x_0 being an appropriate reference point, it is possible to give an explicit form of the decomposition

$$A^\mu(x) = A_{phys}^\mu(x) + A_{pure}^\mu(x), \quad (59)$$

where

$$A_{phys}^\mu(x) = - \int_{x_0}^x \mathcal{W}_C(x, s) F_{\alpha\beta}(s) \mathcal{W}_C(s, x) \frac{\partial s^\alpha}{\partial x_\mu} ds^\beta, \quad (60)$$

$$A_{pure}^\mu(x) = \frac{i}{g} \mathcal{W}(x, x_0) \frac{\partial}{\partial x_\mu} \mathcal{W}_C(x_0, x). \quad (61)$$

Changing the path C alters both of $A_{phys}^\mu(x)$ and $A_{pure}^\mu(x)$, but the sum of them is intact. According to Lorcé, the freedom in the choice of the path is therefore an origin of the Stückelberg symmetry. Accordingly, Stückelberg-invariant quantities must be path-independent, whereas path-dependent quantities are called Stückelberg non-invariant. Based on these considerations, Lorcé comes to conclude that one should distinguish the two forms of gauge-invariance :

- *Strong gauge-invariance* : Quantities with this invariance are invariant under Stückelberg transformation as well as under the usual gauge transformation. They are local quantities and can be measured without relying upon any expansion framework like the twist-expansion in perturbative QCD.
- *Weak gauge-invariance* : Quantities with this invariance are invariant under the usual gauge transformation, but it is not Stückelberg invariant. They are generally nonlocal. They correspond to “quasi-observables”, which can be measured

only within an certain expansion framework. Putting it in another way, they are theoretical-scheme dependent (quasi-)observables.

All these statements might be correct in the most general context. Still, what is lacking in this general argument with highly mathematical nature is the insight into the physical meaning of the decomposition $A_\mu(x) = A_\mu^{phys}(x) + A_\mu^{pure}(x)$. We know that, among the four components of the gauge field $A_\mu(x)$ ($\mu = 0, 1, 2, 3$), independent dynamical degrees of freedom are only two.⁸⁸ They are two transverse components, say A^1 and A^2 . (The other two components, i.e. the so-called scalar component A^0 and longitudinal component A^3 , are not independent dynamical degrees of freedom.) Undoubtedly, what should be identified with the *physical* components of $A_\mu(x)$ are these two transverse components. Unfortunately, as repeatedly emphasized, a delicacy here is the fact that the notion of transversality depends on the Lorentz frame. The standard transverse-longitudinal decomposition $\mathbf{A}(x) = \mathbf{A}_\perp(x) + \mathbf{A}_\parallel(x)$ in the noncovariant framework has a meaning only after specifying a working Lorentz-frame of reference. Nevertheless, it is important to recognize the fact that one has a freedom to start this noncovariant decomposition in an arbitrarily chosen Lorentz frame. The question is therefore whether a quantity we are discussing is a Lorentz-frame-dependent quantity or not. If the quantity of question is a Lorentz-frame-independent one, there is no reason to suspect that the noncovariant treatment as proposed by Chen et al. would give a wrong answer.

Returning to our nucleon spin decomposition problem, a possible candidate of Lorentz-invariant spin observable is helicity.^f This suggests that the nucleon helicity sum rule or the longitudinal nucleon spin decomposition can be made so as to meet both the requirements of gauge-invariance and Lorentz-invariance. However, this conversely indicates that one cannot expect the same for more general nucleon spin decomposition like the decomposition of the transverse nucleon spin. At any rate, when discussing the possibility of gauge-invariant decomposition of the nucleon spin, one should pay more attention to the significant difference between the above two cases. Undoubtedly, what makes our problem difficult is an *intricate interplay* between the gauge- and Lorentz-frame dependencies of the nucleon spin decomposition.

Before ending this section, we think it useful to explain the reason why a lot of researchers believe that the Chen decomposition^{28,29} and the Bashinsky-Jaffe decomposition²⁵ (or Hatta decomposition⁵⁶) are physically inequivalent GIEs of the Jaffe-Manohar decomposition.²¹ The reason can be traced back to Chen et al.'s calculations for the evolution matrix of the quark and gluon longitudinal momentum fraction²⁹ and also for quark and gluon longitudinal spin based on their noncovariant decomposition.³⁴ For example, their prediction for the asymptotic value of the gluon

^fThe helicity is exactly Lorentz-invariant only for a massless particle, but its invariance holds also for a massive particle like the nucleon for a wide-class of Lorentz transformations, which are relevant for our later discussion.

momentum fraction in the nucleon

$$\lim_{Q^2 \rightarrow \infty} \langle x \rangle^G = \frac{8}{8 + 6 n_f} \sim \frac{1}{5}, \quad (62)$$

is drastically different from the standardly-believed value :

$$\lim_{Q^2 \rightarrow \infty} \langle x \rangle^G = \frac{16}{16 + 3 n_f} \sim \frac{1}{2}, \quad (63)$$

which can properly be reproduced in the light-cone gauge calculation. It was conjectured in Ref. 45 and 46 that the calculation of the evolution matrices in the Coulomb gauge (or in the Chen decomposition) is highly nontrivial, and that Chen et al's calculation is probably wrong. This conjecture was, in a sense, explicitly confirmed by the recent study by Ji, Zhang, and Zhao.⁷⁶ For simplicity, we explain the point in simpler Abelian case (QED). They start with the definition of gauge-invariant photon spin *a la* Chen et al.

$$\mathbf{S}_\gamma = \mathbf{E} \times \mathbf{A}_\perp, \quad (64)$$

where \mathbf{A}_\perp is the physical or transverse component of the photon field given as

$$\mathbf{A}_\perp^i = \left(\delta^{ij} - \frac{\nabla_i \nabla_j}{\nabla^2} \right) \mathbf{A}^j. \quad (65)$$

Now, consider a boost along the negative 3-axis with infinite velocity. In this IMF limit, they found that \mathbf{S}_γ takes the following form :

$$\mathbf{S}_\gamma = \mathbf{E} \times \mathbf{A}_\perp \rightarrow \mathbf{E} \times \left(\mathbf{A} - \frac{1}{\nabla^+} \nabla A^+ \right). \quad (66)$$

Here, the quantity

$$\mathbf{A}_{phys} = \mathbf{A} - \frac{1}{\nabla^+} \nabla A^+ \quad (67)$$

is basically the physical component of the photon in the light-cone gauge.^g A delicacy here is that the IMF limit and the loop-integrals necessary for the calculation of the evolution matrix are *noncommutable*. They argue that, if one properly takes care of this noncommutativity, the Chen decomposition gives exactly the same answer as that in the light-cone gauge. This is nice, but there still remains a delicate question. In their whole analysis, they must eventually take the IMF limit. In this sense, the IMF limit (and the light-cone gauge) still play uninterchangeable role ! On the other hand, however, one usually believes that the longitudinal gluon spin, or more generally the longitudinal gluon spin distribution in the nucleon, is a Lorentz-frame-independent quantity, even though its physical interpretation becomes simplest in the IMF. Is ΔG not only gauge-invariant but also Lorentz-frame independent observable ? It appears that their analysis has not given a completely satisfactory answer to this question.

^gIt is important to recognize that this \mathbf{A}_{phys} is nonlocal, but it is nevertheless path-independent. These two types of nonlocality should be clearly distinguished. See the discussion in Ref. 46, for more detail.

4. Recent Controversies on Transverse Nucleon Spin Decomposition

By historical as well as practical reason, our main interest so far has been devoted to the problem of longitudinal nucleon spin sum rule. Naturally, one expects to get useful complimentary information from studies of transverse spin sum rule, which has in fact been a object of intense debate in a few years. The importance of comparing the transverse and longitudinal nucleon spin decomposition in the study of relativistic nucleon spin observables was first emphasized in the paper by Bakker, Leader and Trueman (BLT).²⁷ They proposed a transverse spin sum rule which contains the contribution from the quark transversity distributions. However, as criticized by Ji, Xiong, and Yuan,⁷⁵ since the quark transversity is a chiral-odd object, the proposed sum rule appears in direct contradiction with the chiral-even property of the nucleon spin and orbital angular momentum. Later, Leader proposed another transverse spin sum rule⁴⁸ based on the manipulation in the BLT paper.²⁷ In the case of a transversely polarized nucleon, moving along the positive z axis, he obtains the sum rule :

$$J_q^\perp = \frac{1}{2M} \left[P_0 \int_{-1}^1 x E_q(x, 0, 0) dx + M \int_{-1}^1 x H_q(x, 0, 0) dx \right], \quad (68)$$

where P_0 is the energy of the nucleon. Undoubtedly, this is a Lorentz-frame-dependent sum rule. The reason is simple. This sum rule is obtained by taking a nucleon matrix element of the angular momentum operator J_q^x , which is not invariant under the Lorentz-boost in the z direction, i.e. in the direction along which the nucleon is moving.

With the intension of obtaining a boost-invariant transverse spin sum rule for a moving nucleon (along the z direction), Ji, Xiong, and Yuan attempted to construct the transverse spin sum rule based on the Lorentz-covariant Pauli-Lubanski vector.^{75,89} Their sum rule is given as

$$J_q^\perp = \frac{1}{2} \left[\int_{-1}^1 x E_q(x, 0, 0) dx + \int_{-1}^1 x H_q(x, 0, 0) dx \right], \quad (69)$$

which just takes the same form as the corresponding longitudinal nucleon spin sum rule (the well-known Ji sum rule²²) given as

$$J_q^\parallel = \frac{1}{2} \left[\int_{-1}^1 x E_q(x, 0, 0) dx + \int_{-1}^1 x H_q(x, 0, 0) dx \right], \quad (70)$$

and it appears to be Lorentz-frame-independent. However, a flaw of their derivation was pointed out by several authors.^{49,50,59,80,81} Their derivation is based on the parametrization of the nucleon matrix element of the quark and gluon energy momentum tensor :

$$\begin{aligned} \langle P' S | T_i^{\mu\nu}(0) | P S \rangle = & \bar{U}(P') \left[A_i(\Delta^2) \gamma^{\{\mu} \bar{P}^{\nu\}} + B_i(\Delta^2) \frac{\bar{P}^{\{\mu} i \sigma^{\nu\}} \Delta_\alpha}{2M} \right. \\ & \left. + C_i(\Delta^2) \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2}{M} + \bar{C}_i(\Delta^2) M g^{\mu\nu} \right] U(P), \quad (71) \end{aligned}$$

where $\bar{P} = (P' + P)/2$, $\Delta = P' - P$, and A_i, B_i, C_i and C'_i are generalized form factors, with i denoting either of quark or gluon. In deriving the above sum rule, they erroneously dropped the possible contributions from the \bar{C}_i terms. Later, several authors rederived the transverse nucleon spin sum rule, by careful account of the \bar{C}_i terms.^{50,59,81} However, a worry is that their answers turned out to be totally diverging. For instance, Leader arrived at the answer⁵⁰

$$J_{q/G}^\perp = \frac{1}{2} (A_{q/G}(0) + B_{q/G}(0)) + \frac{P^0 - M}{2P^0} \bar{C}_{q/G}(0), \quad (72)$$

with the relation

$$A_{q/G}(0) = \int_{-1}^1 x H_{q/G}(x, 0, 0) dx,$$

$$B_{q/G}(0) = \int_{-1}^1 x E_{q/G}(x, 0, 0) dx.$$

On the other hand, Hatta, Yoshida, and Tanaka (HYT) gave⁵⁹

$$J_{q/G}^\perp = \frac{1}{2} (A_{q/G}(0) + B_{q/G}(0)) + \frac{P^3}{2(P^0 + M)} \bar{C}_{q/G}(0). \quad (73)$$

Finally, based on the framework of light-front quantization scheme, Harrindranath, Kundu, and Mukerjee (HKM) arrived at the following answer,⁸¹

$$J_{q/G}^\perp = \frac{1}{2} (A_{q/G}(0) + B_{q/G}(0) + \bar{C}_{q/G}(0)). \quad (74)$$

Several comments are in order here. Because of the relation,

$$\bar{C}_q(0) + \bar{C}_G(0) = 0, \quad (75)$$

all these three results are consistent with the net transverse nucleon spin sum rule :

$$J_q^\perp + J_G^\perp = \frac{1}{2} [A_q(0) + B_q(0) + A_G(0) + B_G(0)] = \frac{1}{2}. \quad (76)$$

One also observes that all these three expressions coincide in the infinite-momentum-frame (IMF) limit $P^2 \rightarrow \infty$, $P^0 \rightarrow \infty$. Note, however, that the first two sum rules are generally dependent on the nucleon momentum or energy, so that they are obviously Lorentz-frame-dependent.

Table 1. The various choices of angular momentum tensor and nucleon spinors for obtaining transverse spin sum rule.

	$J^{\alpha\beta}$	$ PS^x\rangle$
Leader	$\int d^3x M^{0\alpha\beta}$	Dirac spinors
HTY	$\int dx^- d^2x^\perp M^{+\alpha\beta}$	Dirac spinors
HKM	$\int dx^- d^2x^\perp M^{+\alpha\beta}$	Light-front spinors

How can we understand these differences ? The reason is that they all calculated the nucleon matrix element of the Pauli-Lubanski vector W^x between the transversely polarized nucleon state in the x direction :

$$\langle PS^x | W^x | PS^x \rangle \quad \text{with} \quad W_\mu = -\frac{1}{2} \epsilon_{\mu\alpha\beta\rho} J^{\alpha\beta} P^\rho, \quad (77)$$

but with different angular momentum tensor $J^{\alpha\beta}$ and different nucleon spinors. As summarized in Table 1, Leader uses the angular momentum tensor $M^{0\alpha\beta}$ in the equal-time (ET) formalism together with Dirac spinors. Hatta et al. (HTY) use the angular momentum tensor in the light-front (LF) formalism $M^{+\alpha\beta}$ together with Dirac Spinors. On the other hand, Harindranath et al. (HKM) use the angular momentum tensor in the LF formalism together with the light-front spinors. HKM emphasize that their result based on the LF (light-front) formalism is absolutely Lorentz-frame-independent, but this statement is misleading. It is known that the use of the LF spinors in the LF formalism is equivalent to working in the IMF. In the IMF, however, the dependence on the nucleon longitudinal momentum P^3 is naturally washed out. What HKM have shown is actually the P_\perp -independence of their sum rule.⁸¹

In any case, one now convinces that the transverse spin sum rule is generally Lorentz-frame-dependent due to the existence of the term $\bar{C}(0)$. It is very important to recognize the fact that the existence of plural forms of transverse spin decomposition has nothing to do with our *gauge problem*, because both of J_q and J_G are obviously gauge-invariant. Rather, one can say that the origin of existence of plural forms of transverse spin sum rule is the *relativity* !

In fact, as nicely reviewed in Ref.90, the treatment of spin in relativistic quantum mechanics is far more complicated than in nonrelativistic quantum mechanics. The relevant complication is known to arise from the fact that the sequences of rotationless Lorentz boosts can generate rotations as dictated by the following commutation relation :

$$[K_i, K_j] = -i \epsilon_{ijk} J_k, \quad (78)$$

where K_i ($i = 1, 2, 3$) are the rotationless boost generators and J_i ($i = 1, 2, 3$) are the spatial rotation generators. For a particle with nonzero mass, it is most convenient to define its spin states in its rest frame^h. However, the relativistic spin states (or observables) generally depend both on the frame where the spin is defined and on a set of Lorentz transformations which relates the frame where the spin is defined and a frame where a particle has a definite momentum \mathbf{p} . This in principle induces an infinite numbers of possible choices of spin observables in relativistic quantum mechanics.

Despite the above general statement, it is very important to recognize a remarkable difference between the transverse spin decomposition and the longitudinal one.

^hThe treatment of massless particles requires another care, since there is no rest frame for a massless particle.

In fact, one can easily verify that any of the above-mentioned three choices for $J^{\alpha\beta}$ and the nucleon spinors leads to exactly the same sum rule for the longitudinal nucleon spin,

$$J_{q/G}^{\parallel} = \frac{1}{2} (A_{q/G}(0) + B_{q/G}(0)) \quad (79)$$

which is nothing but the celebrated Ji sum rule. Important lessons learned from these observations are as follows. First, since the Lorentz-frame-independent decomposition of the transverse nucleon spin into the quark and gluon total angular momenta seems to be impossible, we naturally have no chance to get further frame-independent decomposition of the quark and gluon transverse angular momenta into their intrinsic spin and orbital part. In contrast, we still have a possibility to get a gauge- and frame-independent decomposition of the longitudinal J_q and J_G into their intrinsic spin and orbital parts. This is true even for the massless gluon, for which there is no rest frame. In Sect.5, we shall demonstrate how it is possible for easier QED case, i.e. for massless photons.

5. Does the “Canonical Orbital Angular Momentum” Satisfy the SU(2) Commutation Relation ?

Quite a lot of people believe that a greatest advantage of the canonical type decomposition of the nucleon spin is that each term satisfies the angular momentum commutation relation, and that this is of vital importance for natural interpretation of each term as an angular momentum. One can convince below that this is not necessarily true even in the simpler case of Abelian gauge theory. (Here we closely follow the argument given in Ref. 91 and 92. See also the standard textbooks^{93–95} for general discussion.) The fact is that the observability of the photon spin and OAM has little to do with their SU(2) commutation relations.

Let us start with the textbook expression for the total photon angular momentum,

$$\mathbf{J} = \int \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) d^3r. \quad (80)$$

There is no doubt that this expression is manifestly gauge-invariant. As is well-known, after an appropriate choice of the Lorentz-frame, the vector potential \mathbf{A} of the photon can be gauge-invariantly decomposed into the transverse and longitudinal parts as

$$\mathbf{A}(x) = \mathbf{A}_{\perp}(x) + \mathbf{A}_{\parallel}(x) \equiv \mathbf{A}_{phys}(x) + \mathbf{A}_{pure}(x). \quad (81)$$

This gives the corresponding transverse-longitudinal decomposition of the electric field,

$$\mathbf{E} = \mathbf{E}_{\perp} + \mathbf{E}_{\parallel}, \quad (82)$$

with

$$\mathbf{E}_{\perp} = -\frac{\partial \mathbf{A}_{\perp}}{\partial t}, \quad \mathbf{E}_{\parallel} = -\nabla A^0 - \frac{\partial \mathbf{A}_{\parallel}}{\partial t}. \quad (83)$$

Correspondingly, the total photon angular momentum can be decomposed into two parts as

$$\mathbf{J} = \int \mathbf{r} \times (\mathbf{E}_{\parallel} \times \mathbf{B}) d^3r + \int \mathbf{r} \times (\mathbf{E}_{\perp} \times \mathbf{B}) d^3r \quad (84)$$

$$\equiv \mathbf{J}_{long} + \mathbf{J}_{trans}. \quad (85)$$

By using the Gauss law $\nabla \cdot \mathbf{E}_{\parallel} = \rho$, it is easy to show that the longitudinal part \mathbf{J}_{long} , which contains the \mathbf{E}_{\parallel} component, can be rewritten in the form :

$$\mathbf{J}_{long} = \int \rho (\mathbf{r} \times \mathbf{A}_{\perp}) d^3r \equiv \mathbf{L}_{pot}, \quad (86)$$

which is nothing but the “potential angular momentum” in the terminology of Ref. 41. On the other hand, the transverse part \mathbf{J}_{\perp} can further be decomposed into two parts as,

$$\mathbf{J}_{trans} = \int E_{\perp}^l (\mathbf{r} \times \nabla) A_{\perp}^l d^3r + \int \mathbf{E}_{\perp} \times \mathbf{A}_{\perp} d^3r = \mathbf{L} + \mathbf{S}. \quad (87)$$

which can be identified with the “canonical” OAM and the intrinsic spin of the photon. We emphasize that this decomposition is gauge-invariant, because \mathbf{A}_{\perp} is gauge-invariant (\mathbf{E}_{\perp} is naturally gauge-invariant). We also recall the fact that the sum of the potential angular momentum $\mathbf{J}_{long} = \mathbf{L}_{pot}$ and the “canonical” OAM $\mathbf{L}_{can}^{\gamma} \equiv \int d^3r E_{\perp}^l (\mathbf{r} \times \nabla) A_{\perp}^l$ can be identified with the “mechanical” OAM $\mathbf{L}_{mech}^{\gamma}$ of the *photon*. Note, however, that, $\mathbf{L}_{pot} = 0$ for free photons (since $\rho = 0$), so that in this case, there is no difference between the “mechanical” and “canonical” OAMs. This is the situation which we shall consider below, since very weak interactions between photons and charged particles is allowed to be introduced only at the final stage as a perturbation.

To proceed, we first introduce transverse mode functions with polarization λ as solutions of Helmholtz equation (it is nothing but the Maxwell equation for a free photon) with the transversality condition :

$$\nabla^2 \mathbf{F}_{\lambda} = -k^2 \mathbf{F}_{\lambda}, \quad \nabla \cdot \mathbf{F}_{\lambda} = 0. \quad (88)$$

They are supposed to satisfy the following orthonormalization condition :

$$\langle \mathbf{F}_{\lambda} | \mathbf{F}_{\lambda'} \rangle \equiv \int \mathbf{F}_{\lambda} \cdot \mathbf{F}_{\lambda'} d^3r = \delta_{\lambda\lambda'}. \quad (89)$$

The simplest choice for the transverse mode functions would be the circularly polarized plane waves :

$$\mathbf{F}_{\lambda} = \frac{1}{\sqrt{V}} \boldsymbol{\varepsilon}_{\mathbf{k},s} e^{i\mathbf{k} \cdot \mathbf{r}} \quad (s = \pm 1), \quad (90)$$

but we can also take other choices like that of the para-axial laser beams used in the measurements of the photon orbital angular momentum.^{96–98} With these mode functions, the electromagnetic field or vector potential can be expanded as

$$\mathbf{A}_{\perp} = \sum_{\lambda} \sqrt{\frac{\hbar}{2\omega_{\lambda}}} \left[a_{\lambda} \mathbf{F}_{\lambda} + a_{\lambda}^{\dagger} \mathbf{F}_{\lambda}^* \right], \quad (91)$$

where a_λ and a_λ^\dagger are the annihilation and creation operator of the photon with the polarization λ , satisfying the standard commutation relation :

$$[a_\lambda, a_{\lambda'}^\dagger] = \delta_{\lambda\lambda'}. \quad (92)$$

The corresponding mode-expansions for the electric and magnetic fields are then given by

$$\mathbf{E}_\perp = \sum_\lambda i \sqrt{\frac{\hbar \omega_\lambda}{2}} \left[a_\lambda \mathbf{F}_\lambda - a_\lambda^\dagger \mathbf{F}_\lambda^* \right], \quad (93)$$

$$\mathbf{B}_\perp = \sum_\lambda i \sqrt{\frac{\hbar}{2 \omega_\lambda}} \left[a_\lambda \nabla \times \mathbf{F}_\lambda + a_\lambda^\dagger \nabla \times \mathbf{F}_\lambda^* \right]. \quad (94)$$

Using these formulas, we are led to the second-quantized forms of the intrinsic spin and “canonical” OAM of the photon as

$$\mathbf{S} \equiv \int \mathbf{E}_\perp \times \mathbf{A}_\perp d^3r = \frac{1}{2} \sum_{\lambda, \lambda'} [a_\lambda^\dagger a_{\lambda'} + a_{\lambda'} a_\lambda^\dagger] \langle \mathbf{F}_\lambda | \hat{\mathbf{S}} | \mathbf{F}_{\lambda'} \rangle, \quad (95)$$

$$\mathbf{L} \equiv \int \mathbf{E}_\perp^l (\mathbf{r} \times \nabla) \mathbf{A}_\perp^l d^3r = \frac{1}{2} \sum_{\lambda, \lambda'} [a_\lambda^\dagger a_{\lambda'} + a_{\lambda'} a_\lambda^\dagger] \langle \mathbf{F}_\lambda | \hat{\mathbf{L}} | \mathbf{F}_{\lambda'} \rangle, \quad (96)$$

with the definitions of the operators $\hat{\mathbf{S}}$ and $\hat{\mathbf{L}}$:

$$(\hat{\mathbf{S}})_{ij} = -i \hbar \varepsilon_{ijk}, \quad \hat{\mathbf{L}} = -i \hbar (\mathbf{r} \times \nabla) \quad (97)$$

These operators $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ certainly satisfy the familiar SU(2) algebra :

$$[\hat{S}_i, \hat{S}_j] = i \hbar \hat{S}_k, \quad [\hat{L}_i, \hat{L}_j] = i \hbar \hat{L}_k. \quad (98)$$

However, the crucial point here is that what correspond to observables are not $\hat{\mathbf{S}}$ and $\hat{\mathbf{L}}$ but \mathbf{S} and \mathbf{L} , because the latter are operators acting on *physical Fock space*. What are the commutation relations of \mathbf{S} and \mathbf{L} like, then ? To find them, choose circularly polarized plane waves again as field modes,

$$\mathbf{F}_\lambda = \frac{1}{\sqrt{V}} \boldsymbol{\varepsilon}_{\mathbf{k},s} e^{i \mathbf{k} \cdot \mathbf{r}} \quad (s = \pm 1). \quad (99)$$

In this case, \mathbf{S} is represented as

$$\mathbf{S} = \sum_{\mathbf{k}} \frac{\hbar \mathbf{k}}{k} (a_{\mathbf{k},1}^\dagger a_{\mathbf{k},1} - a_{\mathbf{k},-1}^\dagger a_{\mathbf{k},-1}). \quad (100)$$

Note that \mathbf{S} is given as a superposition of the number operators of photons with definite polarizations, so that any components of \mathbf{S} must commute :

$$[S_i, S_j] = 0. \quad (101)$$

Somewhat unexpectedly, we therefore find that \mathbf{S} does not satisfy standard angular momentum commutation relation. This means that \mathbf{S} does not generate general rotations of photon polarization states. Instead, it generates a transformation of the polarization vector such that the transversality of the polarization vector is

preserved. To be more concrete, under a rotation by angle θ about the direction of the photon momentum \mathbf{k} , the photon state with helicity λ transforms as⁹⁹

$$|\mathbf{k}; \lambda\rangle \rightarrow \exp \left[-i\theta \frac{\mathbf{S} \cdot \mathbf{k}}{|\mathbf{k}|} \right] |\mathbf{k}; \lambda\rangle = e^{-i\theta\lambda} |\mathbf{k}; \lambda\rangle. \quad (102)$$

This shows that only the components of the operator \mathbf{S} along \mathbf{k} , i.e. the (the propagation direction of the photon) is a true spin angular momentum operator in the sense that this component certainly generate spin rotation. The components of the operator \mathbf{S} along \mathbf{k} is nothing but the *helicity* of the photon.

What about the commutation relation of \mathbf{L} , then? First, notice that the total photon angular momentum must satisfy the standard commutation relation,

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k. \quad (103)$$

Second, \mathbf{S} and \mathbf{L} must transform as vectors under spatial rotation, so that

$$[J_i, S_j] = i\hbar \varepsilon_{ijk} S_k \quad (104)$$

$$[J_i, L_j] = i\hbar \varepsilon_{ijk} L_k. \quad (105)$$

Combining these relations with the commutation relation $[S_i, S_j] = 0$ for \mathbf{S} , it follows that

$$[L_i, L_j] = i\hbar \varepsilon_{ijk} (L_k - S_k) \quad (106)$$

$$[L_i, S_j] = i\hbar \varepsilon_{ijk} S_k \quad (107)$$

Thus, one clearly sees that \mathbf{L} does not satisfy the standard angular momentum algebra either, even though it is the very quantity, which can be measured, for instance, as OAM of para-axial laser beam.^{96–98} We therefore confirm that, for massless particles like photons (and naturally also for gluons), there is no connection between the observability and the requirement of the SU(2) commutation relation of each piece of spin and orbital angular momentum decomposition. All these delicacies of photon (or gluon) spin decomposition comes from the fact that there is no rest frame for massless particles! In a mathematical language, as emphasized by Zhang and Pak,⁶⁶ the only frame-independent notion of spin for a massless particle is the *helicity*, which can be described by a little group $E(2)$ of the Lorentz group.

Here we do not go further into the detail. But, by introducing the interaction of the photon beam with atoms, the following conclusion can be drawn. (Interested readers are recommended to consult with the original papers.^{91,92}) Both spin \mathbf{S} and orbital angular momentum \mathbf{L} of a photon are well defined quantities and might in principle separately be measured. However, in practice, only the components along the propagation direction can be measured by detecting the change in internal and external angular momentum of atoms. This indicates that, also in the problem of complete decomposition of the nucleon spin, only the longitudinal spin decomposition or the helicity sum rule (not the transverse spin decomposition) would be related to direct observables.

6. What is “Potential Angular Momentum” ?

We have already pointed out that the difference between the two decompositions of the nucleon spin is characterized by the two physically inequivalent orbital angular momenta, i.e. the generalized “canonical” OAM and the “mechanical” OAM. Because the difference of these two OAMs is characterized by the “potential angular momentum” term,⁴¹ a clear understanding of it is very important for answering the following question, i.e. “Which of the above two OAMs can be thought as more *physical* from the observational viewpoint ?”

Let us first recall the relation,

$$\mathbf{L}_{can} = \mathbf{L}_{mech} + \mathbf{L}_{pot}, \quad (108)$$

with

$$\mathbf{L}_{can} = \int \psi^\dagger \mathbf{r} \times \left(\frac{1}{i} \nabla - g \mathbf{A}_{pure} \right) \psi d^3r, \quad (109)$$

$$\mathbf{L}_{mech} = \int \psi^\dagger \mathbf{r} \times \left(\frac{1}{i} \nabla - g \mathbf{A} \right) \psi d^3r, \quad (110)$$

$$\mathbf{L}_{pot} = g \int \psi^\dagger \mathbf{r} \times \mathbf{A}_{phys} \psi d^3r, \quad (111)$$

which means that the gauge-invariant canonical OAM is obtained as a sum of the mechanical OAM and the potential OAM defined in Ref. 41. This is clearly different from the definition adopted by Hatta and Yoshida,⁵⁸ which is given by

$$\mathbf{L}_{mech} = \mathbf{L}_{can} + \mathbf{L}'_{pot}, \quad (112)$$

with

$$\mathbf{L}'_{pot} = -g \int \psi^\dagger \mathbf{r} \times \mathbf{A}_{phys} \psi d^3r. \quad (113)$$

One might think that it is just a matter of sign convention of \mathbf{L}_{pot} term. It would certainly be so if one is interested only in the difference between the two OAMs, i.e. the canonical and the mechanical OAMs. However, if one is interested in the separate physical contents of the two OAMs, one will find that these two definitions have quite different physical interpretations. In fact, as we shall see in the next section, proper physical interpretation of the two OAMs has a deep connection with the problem of practical observability of the OAMs through deep-inelastic-scattering (DIS) measurements.

The reason of Hatta and Yoshida’s definition of \mathbf{L}'_{pot} can readily be imagined from the following consideration. In view of the fact that the potential angular momentum contains the physical component \mathbf{A}_{phys} of the gluon field, \mathbf{L}'_{pot} is naturally thought to give a measure of the genuine quark-gluon interaction. In fact, they showed that \mathbf{L}'_{pot} is related to the twist-3 quark-gluon correlation functions. The spirit of their definition (112) with (113) would then be the following. Formally, the expression (110) of the mechanical OAM \mathbf{L}_{mech} contains the full gluon field \mathbf{A} . Then, the subtraction of \mathbf{L}'_{pot} from \mathbf{L}_{mech} would work to eliminate the physical

part \mathbf{A}_{phys} of the gluon field, thereby leading to the generalized (gauge-invariant) canonical OAM \mathbf{L}_{can} , in which only the pure-gauge part \mathbf{A}_{pure} of the gluon is contained. The resultant \mathbf{L}_{can} is essentially the standard canonical OAM, since \mathbf{A}_{pure} is an unphysical gauge degrees of freedom, which can be eliminated eventually. It also appears that this canonical OAM is perfectly consistent with free partonic picture of quark orbital motion, as already emphasized in the paper by Bashinsky and Jaffe.²⁵ In this picture, what contains the genuine quark-gluon interaction is \mathbf{L}_{mech} not \mathbf{L}_{can} .

However, this viewpoint is not necessarily justified. Another totally different viewpoint would be the following. One takes that \mathbf{L}_{mech} is a quantity with more physical significance than \mathbf{L}_{can} . In fact, what appears in the equation of motion of the charged particle under the presence of the electromagnetic potential is the mechanical momentum \mathbf{P}_{mech} and the mechanical OAM \mathbf{L}_{mech} not the canonical momentum \mathbf{P}_{can} and the canonical OAM \mathbf{L}_{can} .¹⁰⁰ Remember that the mechanical OAM is a quantity which is related to the the coordinate and the velocity of a particle as $\frac{m\mathbf{r}\times\mathbf{u}}{\sqrt{1-u^2}}$ with $\mathbf{u} = \dot{\mathbf{r}}$. To understand the physical meaning of the mechanical OAM, the relativistic kinematics of the charged particle (electron) is not essential. It would rather block up transparent understanding of the physical meaning of the mechanical OAM as well as the relation between the mechanical OAM and canonical OAM. In the following, we therefore consider simpler interacting system of photons and charged particles with nonrelativistic motion⁹³⁻⁹⁵ following the discussion in Ref. 45. The spin of the charged particle is also discarded, for simplicity.

The total energy of such a system is given by

$$H = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 + \frac{1}{2} \int [\mathbf{E}^2 + \mathbf{B}^2] d^3r. \quad (114)$$

Here the 1st and the 2nd terms of the r.h.s. respectively stand for the mechanical kinetic energy of the charged particles and the total energy of the electromagnetic fields.

Introducing the transverse-longitudinal decomposition $\mathbf{A}(x) = \mathbf{A}_\perp(x) + \mathbf{A}_\parallel(x)$ of the vector potential, the electric field can also be decomposed into longitudinal and transverse components as

$$\mathbf{E} = \mathbf{E}_\perp + \mathbf{E}_\parallel, \quad (115)$$

with

$$\mathbf{E}_\perp = -\frac{\partial \mathbf{A}_\perp}{\partial t}, \quad \mathbf{E}_\parallel = -\nabla A^0 - \frac{\partial \mathbf{A}_\parallel}{\partial t}, \quad (116)$$

while the magnetic field is intrinsically transverse

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}_\perp = \mathbf{B}_\perp. \quad (117)$$

Correspondingly, the photon part of the total energy can be decomposed into two

pieces, i.e. the longitudinal part and the transverse part, as

$$H = \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 + \frac{1}{2} \int \mathbf{E}_{\parallel}^2 d^3r + \frac{1}{2} \int [\mathbf{E}_{\perp}^2 + \mathbf{B}_{\perp}^2] d^3r. \quad (118)$$

By using the Gauss law $\nabla \cdot \mathbf{E}_{\parallel} = \rho$, it can be shown that the longitudinal part is nothing but the Coulomb energy between the charged particles (aside from the self-energies), so that we can write as

$$H = \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 + V_{coul} + H_{trans}, \quad (119)$$

with

$$V_{Coul} = \frac{1}{4\pi} \sum_{i,j=1}^N \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (120)$$

$$H_{trans} = \frac{1}{2} \int [\mathbf{E}_{\perp}^2 + \mathbf{B}_{\perp}^2] d^3r. \quad (121)$$

Next, let us consider a similar decomposition of the total momentum. The total momentum of the system is a sum of the mechanical momentum of charged particles and the momentum of the photon field as

$$\mathbf{P} = \mathbf{P}_{mech} + \mathbf{P}^{\gamma} \quad (122)$$

with

$$\mathbf{P}_{mech} = \sum_i m_i \dot{\mathbf{r}}_i, \quad \mathbf{P}^{\gamma} = \int \mathbf{E} \times \mathbf{B} d^3r. \quad (123)$$

The total momentum of the electromagnetic fields can be decomposed into longitudinal and transverse parts as

$$\mathbf{P}^{\gamma} = \mathbf{P}_{long}^{\gamma} + \mathbf{P}_{trans}^{\gamma}, \quad (124)$$

with

$$\mathbf{P}_{long}^{\gamma} = \int \mathbf{E}_{\parallel} \times \mathbf{B}_{\perp} d^3r, \quad (125)$$

$$\mathbf{P}_{trans}^{\gamma} = \int \mathbf{E}_{\perp} \times \mathbf{B}_{\perp} d^3r, \quad (126)$$

Again, by using the Gauss law, it can be shown that the longitudinal part \mathbf{P}_{long} is also expressed as

$$\mathbf{P}_{long}^{\gamma} = \sum_i q_i \mathbf{A}_{\perp}(\mathbf{r}_i) \equiv \mathbf{P}_{pot}. \quad (127)$$

As pointed out in Ref. 41, the quantity $q_i \mathbf{A}_{\perp}(\mathbf{r}_i)$ appearing here is nothing but the *potential momentum* according to the terminology of Konopinski.¹⁰¹ In the present context, it represents the momentum that associates with the longitudinal (electric)

field generated by the particle i . After these steps, the total momentum of the system is now given as a sum of three terms as

$$\mathbf{P} = \mathbf{P}_{mech} + \mathbf{P}_{pot} + \mathbf{P}_{trans}^\gamma. \quad (128)$$

A delicate question here is the following. Which of particles or photons should the potential momentum be attributed to? In view of the fact that the potential momentum term is also thought of as representing the interactions of charged particles and photons, we realize that it is of the same sort of question as which of charged particles or photons should the Coulomb energy be attributed to. To attribute it to charged particle is closer to the concept of “action at a distance theory”, while to attribute it to electromagnetic field is closer to the concept of “action through medium”. If there is no difference between their physical predictions, the choice is certainly a matter of convenience. Let us see what happens if we combine the potential momentum term with the mechanical momentum of charged particles. To this end, we recall that, under the presence of electromagnetic potential, the canonical momentum \mathbf{p}_i of the charged particle i is given by the equation

$$\mathbf{p}_i \equiv \frac{\partial L}{\partial \dot{\mathbf{r}}_i} = m_i \dot{\mathbf{r}}_i + q_i \mathbf{A}(\mathbf{r}_i), \quad (129)$$

where L is the Lagrangian corresponding to the Hamiltonian (114). Using it, the total momentum \mathbf{P} can be expressed in the following form :

$$\mathbf{P} = \mathbf{P}_{can} + \mathbf{P}_{trans}^\gamma, \quad (130)$$

with

$$\mathbf{P}_{can} = \sum_i (\mathbf{p}_i - q_i \mathbf{A}_\parallel(\mathbf{r}_i)). \quad (131)$$

Here use has been made of the relation $\mathbf{A}(\mathbf{r}_i) - \mathbf{A}_\perp(\mathbf{r}_i) = \mathbf{A}_\parallel(\mathbf{r}_i)$. Note that the above \mathbf{P}_{can} is the generalized (gauge-invariant) canonical momentum of the charged particle system.

We can carry out a similar manipulation also for the angular momentum. The total angular momentum of the system is a sum of the mechanical angular momentum of charged particles and the angular momentum of photon fields as

$$\mathbf{J} = \mathbf{L}_{mech} + \mathbf{J}^\gamma, \quad (132)$$

with

$$\mathbf{L}_{mech} = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i, \quad (133)$$

$$\mathbf{J}^\gamma = \int d^3r \, \mathbf{r} \times (\mathbf{E} \times \mathbf{B}). \quad (134)$$

Similarly as before, the total angular momentum of the electromagnetic fields can be decomposed into longitudinal and transverse parts as

$$\mathbf{J}^\gamma = \mathbf{J}_{long}^\gamma + \mathbf{J}_{trans}^\gamma, \quad (135)$$

with

$$\mathbf{J}_{long}^\gamma = \int \mathbf{r} \times (\mathbf{E}_\parallel \times \mathbf{B}_\perp) d^3r, \quad (136)$$

$$\mathbf{J}_{trans}^\gamma = \int \mathbf{r} \times (\mathbf{E}_\perp \times \mathbf{B}_\perp) d^3r. \quad (137)$$

Again, by using the Gauss law, \mathbf{J}_{long} can also be expressed as

$$\mathbf{J}_{long}^\gamma = \sum_i q_i \mathbf{r}_i \times \mathbf{A}_\perp(\mathbf{r}_i) \equiv \mathbf{L}_{pot}. \quad (138)$$

This is nothing but the potential angular momentum in the terminology of Ref. 41. Thus, we are led to the following decomposition of the total angular momentum :

$$\mathbf{J} = \mathbf{L}_{mech} + \mathbf{L}_{pot} + \mathbf{J}_{trans}^\gamma. \quad (139)$$

Again, we have freedom to combine the potential angular momentum with the mechanical angular momentum of the charged particle i . Noting again the relation $\mathbf{A}(\mathbf{r}_i) - \mathbf{A}_\perp(\mathbf{r}_i) = \mathbf{A}_\parallel(\mathbf{r}_i)$, we obtain

$$\mathbf{J} = \mathbf{L}_{can} + \mathbf{J}_{trans}^\gamma, \quad (140)$$

with

$$\mathbf{L}_{can} = \sum_i \mathbf{r}_i \times (\mathbf{p}_i - q_i \mathbf{A}_\parallel(\mathbf{r}_i)). \quad (141)$$

As expected, this \mathbf{L}_{can} is just the generalized (gauge-invariant) canonical OAM of the charged particles.ⁱ At first sight, simpler-looking appearance of the decomposition (130) of the total momentum and the decomposition (140) of the total angular momentum appears to indicate physical superiority of canonical momentum and the canonical angular momentum over the mechanical ones. In fact, since $\mathbf{A}_\parallel(\mathbf{r}_i)$ is the pure-gauge part of the photon, which can eventually be eliminated, they are essentially the momentum and the angular momentum of a free particle. However, one should recognize the fact that, under the circumstance where strong electromagnetic potential exists, there cannot be any free charged particle. (This observation becomes of more practical importance in the strong-coupled gauge theory like QCD.) As is clear from the expressions,

$$\mathbf{P}_{mech} = \sum_i m_i \dot{\mathbf{r}}_i = \sum_i m_i \mathbf{v}_i, \quad (142)$$

$$\mathbf{L}_{mech} = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i, \quad (143)$$

what have natural interpretation as translational and orbital motions of particles under the presence of the gauge potential are the mechanical momentum \mathbf{P}_{mech}

ⁱThis precisely corresponds to the gauge-invariant canonical OAM appearing in the Chen decomposition. As is obvious from the above derivation, the gauge degrees of freedom carried by \mathbf{A}_\parallel is not introduced by the artificial prescription of gauge-invariant extension. Rather, it is a freedom already existing in the original gauge theory, i.e. in QED.

and the mechanical OAM \mathbf{L}_{mech} not the canonical momentum \mathbf{P}_{can} and the canonical OAM \mathbf{L}_{can} .¹⁰⁰ It may sound paradoxical, but in conjunction with the relation $\mathbf{P}_{can} = \mathbf{P}_{mech} + \mathbf{P}_{pot}$, and $\mathbf{L}_{can} = \mathbf{L}_{mech} + \mathbf{L}_{pot}$, one must say that what contains extra interaction terms, i.e. the potential momentum and potential angular momentum, are rather the canonical momentum and canonical OAM not the mechanical momentum and the mechanical OAM.

One might still suspect that the argument above is just a matter of philosophy. Naturally, what discriminates physics from philosophy is the experimental observations. In Sect.7, we will show that the above-mentioned difference between the canonical OAM and the mechanical OAM has a crucial influence on their observability by means of high-energy deep-inelastic-scattering measurements.

7. On the Relation with Deep-Inelastic-Scattering Observables

Historically, it was a common belief that the canonical OAMs appearing in the Jaffe-Manohar decomposition would not correspond to observables, because they are not gauge-invariant quantities. This nebulous impression did not change even after a gauge-invariant version of the Jaffe-Manohar decomposition due to Bashinsky and Jaffe or of Chen et al. appeared. However, the situation has changed drastically after Lorcé and Pasquini showed that the canonical quark OAM can be related to a certain moment of a quark distribution function in a phase space, called the Wigner distribution.¹⁰² (A complete classification of the Wigner distributions for a spin 1/2 target is given in the paper by Meissner, Metz, and Schlegel.¹⁰³ See also the paper by Lorcé and Pasquini for an extension to gluon distributions.¹⁰⁴) Since the longitudinal component of the OAM arises from the motion of partons in the transverse plane perpendicular to the nucleon momentum, it is intuitively natural to consider such generalized distribution functions of partons beyond the collinear distributions. The relevant quantity here is a phase-space quark distribution in a longitudinally polarized nucleon :

$$\begin{aligned} \rho^q(x, \mathbf{k}_\perp, \mathbf{b}_\perp; \mathcal{W}) &= \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{-i \Delta_\perp \cdot \mathbf{b}_\perp} \frac{1}{2} \int \frac{dz^- d^2 z_\perp}{(2\pi)^3} e^{i(x \bar{P}^+ z^- - \mathbf{k}_\perp \cdot \mathbf{z}_\perp)} \\ &\times \langle P'^+, \frac{\Delta_\perp}{2}, S | \bar{\psi} \left(-\frac{z}{2} \right) \gamma^+ \mathcal{W} \psi \left(\frac{z}{2} \right) | P^+, -\frac{\Delta_\perp}{2}, S \rangle_{z^+=0}, \end{aligned} \quad (144)$$

given as a function of the ordinary longitudinal momentum fraction x ($x = k^+ / \bar{P}^+$ with $\bar{P} = (P' + P) / 2$), the transverse momentum \mathbf{k}_\perp , the impact parameter \mathbf{b}_\perp . Here, \mathcal{W} is a gauge-link, also called the Wilson line, connecting the two space-time points $z/2$ and $-z/2$. According to them, the Wigner distribution gives a natural definition of the quark OAM density in the phase-space as follows :

$$L_z^q(x, \mathbf{k}_\perp, \mathbf{b}_\perp; \mathcal{W}) = (\mathbf{b}_\perp \times \mathbf{k}_\perp)_z \rho^q(x, \mathbf{k}_\perp, \mathbf{b}_\perp; \mathcal{W}). \quad (145)$$

After integrating over x, \mathbf{k}_\perp , and \mathbf{b}_\perp , they arrive at a remarkable relation, which connects a Wigner distribution with the quark OAM :

$$\begin{aligned} \langle L_z^q \rangle^{\mathcal{W}} &= \int dx d^2 k_\perp d^2 b_\perp L_z^q(x, \mathbf{k}_\perp, \mathbf{b}_\perp; \mathcal{W}) \\ &= - \int dx d^2 k_\perp \frac{\mathbf{k}_\perp^2}{M^2} F_{1,4}^q(x, 0, \mathbf{k}_\perp^2, 0, 0, \mathcal{W}), \end{aligned} \quad (146)$$

where the function $F_{1,4}^q$ is contained in the following structure of the Wigner distribution ρ^q :

$$\begin{aligned} \rho^q(x, \mathbf{k}_\perp, \mathbf{b}_\perp; \mathcal{W}) &= F_{1,1}^q(x, \mathbf{k}_\perp^2, \mathbf{k}_\perp \cdot \mathbf{b}_\perp, \mathbf{b}_\perp^2; \mathcal{W}) \\ &\quad - \frac{1}{M^2} (\mathbf{k}_\perp \times \nabla_{\mathbf{b}_\perp})_z F_{1,4}^q(x, \mathbf{k}_\perp^2, \mathbf{k}_\perp \cdot \mathbf{b}_\perp, \mathbf{b}_\perp^2; \mathcal{W}). \end{aligned} \quad (147)$$

A delicacy here is that the Wigner distribution ρ^q generally turns out to depend on the chosen path of the gauge-link \mathcal{W} connecting the points $z/2$ and $-z/2$. As shown by a careful study by Hatta⁵⁶ with the choice of a staple-like gauge-link in the light-front direction, corresponding to the kinematics of the semi-inclusive reactions or the Drell-Yan processes, the above quark OAM turns out to coincide with the (gauge-invariant) canonical quark OAM not the dynamical OAM :

$$L_{can}^q = \langle L_z^q \rangle^{\mathcal{W}=\mathcal{W}^{LC}}. \quad (148)$$

This observation holds out a hope that the canonical quark OAM in the nucleon would also be a measurable quantity, at least in principle.

In a recent paper,¹⁰⁵ however, Courtoy et al. throws a serious doubt on the practical observability of the Wigner function $F_{1,4}^q$ appearing in the above intriguing sum rule. According to them, even though $F_{1,4}^q$ may be nonzero in particular models and also in real QCD, its observability would be inconsistent with the following observations :

- it drops out in both the formulation of GPDs and TMDs ;
- it is parity-odd, at variance with parity-even structure of more familiar TMD Sivers function^{106,107} ;
- it is nonzero only for imaginary values of the quark-proton helicity amplitudes.

These observations indicate that $F_{1,4}^q$ would not appear in the cross section formulas of any DIS processes at least at the leading order approximation. Anyhow, what is indicated by their arguments is the fact that the existence of a simple partonic picture of the canonical quark OAM in the Fock space and its observability are different things. It appears to us that this takes a discussion on the observability of the canonical OAM back to its starting point.

What about observability of another OAMs, i.e. the mechanical OAMs, then ? As pointed out in Sect.2, we already know the relations connecting the mechanical

quark and gluon OAMs to DIS observables :

$$L_{mech}^q = \frac{1}{2} \int x [H^q(x, 0, 0) + E^q(x, 0, 0)] dx - \frac{1}{2} \int \tilde{H}^q(x, 0, 0) dx, \quad (149)$$

$$L_{mech}^G = \frac{1}{2} \int x [H^G(x, 0, 0) + E^G(x, 0, 0)] dx - \int \tilde{H}^G(x, 0, 0) dx. \quad (150)$$

They are indirect relations, however, in the sense that both the quark and gluon mechanical OAM are obtained only as differences of total angular momenta and the intrinsic spin parts. It would be nicer, if there is any sum rule which directly relates the mechanical OAMs to observables. Fortunately, at least for the quark part, such a relation exists, as first noticed by Penttinen, Polyakov, and Shuvaev, and Strikman¹⁰⁸ and later refined by Kiptily and Polyakov.¹⁰⁹ (The same relation has recently been rediscovered by Hatta and Yoshida in their twist-3 analysis of the nucleon spin contents.⁵⁸) They showed that the mechanical quark OAM L_{mech}^q can be related to a moment of the twist-3 GPD named G_2 as

$$L_{mech}^q = - \int x G_2^q(x, 0, 0) dx. \quad (151)$$

This twist-3 GPD G_2^q appears in the following parametrization of the GPD :

$$\begin{aligned} & \frac{1}{2} \int \frac{dz^-}{2\pi} e^{i x \bar{P}^+ z^-} \langle P', S' | \bar{\psi} \left(-\frac{z^-}{2} \right) \gamma^j \mathcal{W} \psi \left(\frac{z^-}{2} \right) | P, S \rangle \\ &= \frac{1}{2 \bar{P}^+} \bar{u}(P', S') \left[\frac{\Delta_\perp^j}{2 M} G_1^q + \gamma^j (H^q + E^q + G_2^q) \right. \\ & \quad \left. + \frac{\Delta_\perp^j \gamma^+}{P^+} G_3^q + \frac{i \epsilon_T^{jk} \Delta_\perp^k \gamma^+ \gamma_5}{P^+} G_4^q \right] u(P, S), \quad (152) \end{aligned}$$

where $\bar{P} = (P' + P)/2$, the nucleon is moving in the z -direction, Δ_\perp is the transverse part of Δ^μ , and the indices $j, k = 1, 2$ are transverse indices. (We recall that the GPD G_2 has some relation to the quantities \tilde{E}_{2T} and E_+^3 respectively discussed in Ref. 103 and in Ref. 110.) Since the mechanical OAM can be given as a x -integral of the quantity $-x G_2^q(x, 0, 0)$, one might be tempted to interpret the latter as a mechanical OAM density in the Feynman x -space. An interesting observation by Kiptily and Polyakov is the following. According to them, $G_2^q(x, 0, 0)$ consists of the Wandzura-Wilczek (WW) part and the genuine twist-3 part as follows :

$$G_2^q(x, 0, 0) = G_2^{q, WW}(x, 0, 0) + \bar{G}_2^q(x, 0, 0). \quad (153)$$

Here, the WW part is represented by the forward limits of the three twist-2 GPDs as

$$\begin{aligned} G_2^{q, WW}(x, 0, 0) &= -(H^q(x, 0, 0) + E^q(x, 0, 0)) + \frac{1}{x} \tilde{H}^q(x, 0, 0) \\ &+ \int_x^{\epsilon(x)} \frac{dy}{y} (H^q(y, 0, 0) + E^q(y, 0, 0)) - \int_x^{\epsilon(x)} \frac{dy}{y^2} \tilde{H}^q(y, 0, 0), \quad (154) \end{aligned}$$

with $\epsilon(x) = x/|x|$. On the other hand, the 2nd moment of the genuine twist-3 part of G_2^q is shown to vanish identically,

$$\int_{-1}^1 x \bar{G}_2^q(x, 0, 0) dx = 0. \quad (155)$$

This means that the genuine twist-3 part of G_2^q does not contribute at all to the net (or integrated) mechanical quark OAM L_{mech}^q . Putting it in another way, the net mechanical quark OAM is determined solely by three twist-2 GPDs $H^q(x, 0, 0)$, $E^q(x, 0, 0)$, and $\tilde{H}^q(x, 0, 0)$.

Now we recall the discussion in Sect.6 on the relation between the mechanical and canonical OAMs. According to the definition of Hatta and Yoshida,⁵⁸ the mechanical quark OAM is given as a sum of the canonical quark OAM and the potential angular momentum as

$$L_{mech}^q = L_{can}^q + L'_{pot}. \quad (156)$$

As pointed out there, they showed that the potential angular momentum L'_{pot} is related to the ('F-type') twist-3 quark-gluon correlator $\Phi_F(x_1, x_2)$ as

$$L'_{pot} = \int dx_1 dx_2 \mathcal{P} \frac{1}{x_1 - x_2} \Phi_F(x_1, x_2), \quad (157)$$

with \mathcal{P} denoting a principle value, which means that L'_{pot} is a genuine twist-3 quantity. On the other hand, we have seen above that the mechanical quark OAM L_{mech}^q appearing in the left-hand-side of Eq.(156) is given by the twist-2 GPDs alone. This dictates that the genuine twist-3 contribution in L_{can}^q and L'_{pot} must cancel each other in their sum. Is this cancellation accidental? Very curiously, if one takes a different viewpoint as advocated in Sect.6, one can explain the above observation in more natural way. Namely, by translating the QED argument explained in Sect.6 into the QCD problem, one observes that the canonical OAM rather emerges as a sum of the mechanical OAM and the potential angular momentum as

$$L_{can}^q = L_{mech}^q + L_{pot}, \quad (158)$$

with the relation $L_{pot} = -L'_{pot}$. Now it is no surprise that the canonical OAM contains the genuine twist-3 part, since it is given as a sum of the mechanical OAM (given by the twist-2 GPDs alone), and the genuine twist-3 potential angular momentum. We emphasize that this interpretation is in perfect harmony with the statement in sect.6, which tells that what contains the potential angular momentum is the canonical OAM rather than the mechanical OAM. The consideration above, especially the relation (155), also explains naturally why the mechanical OAM rather than the canonical OAM can be considered as the *physical* one, even though it does not appear to fit with the widespread belief that the canonical OAM is more compatible with the partonic interpretation of the OAM angular momentum.

The relation between the two kinds of OAMs was also analysed by Burkardt from a different viewpoint.⁶² His attention is paid to a physical interpretation of the

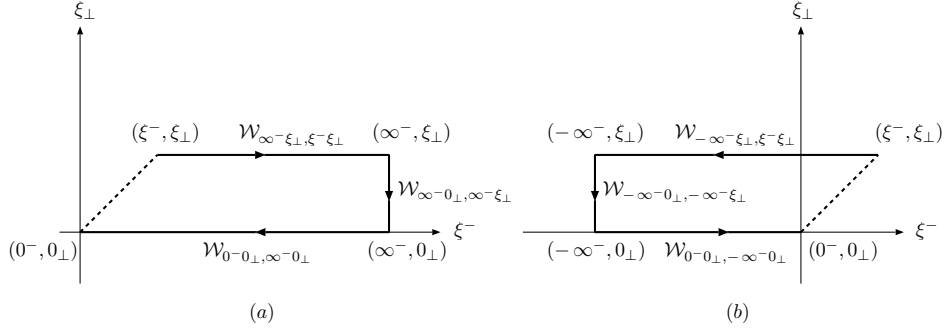


Fig. 3. The future-pointing staple-like Wilson line (left panel) and the past-pointing staple-like Wilson line, corresponding to the kinematics of semi-inclusive hadron productions and the Drell-Yan processes, respectively.

difference between the two OAMs. Also very interesting is his parallel consideration on the difference between several average transverse momenta of quarks inside the nucleon. His analysis begins with the following definitions of the average transverse momenta of quarks and the longitudinal component of the orbital angular momenta in terms of Wigner distributions :

$$\langle \mathbf{k}_\perp^q \rangle^{\mathcal{W}} = \int dx d^2 \mathbf{b}_\perp d^2 \mathbf{k}_\perp \mathbf{k}_\perp \rho^q(x, \mathbf{b}_\perp, \mathbf{k}_\perp; \mathcal{W}), \quad (159)$$

$$\langle L_z^q \rangle^{\mathcal{W}} = \int dx d^2 b_\perp d^2 k_\perp (\mathbf{b}_\perp \times \mathbf{k}_\perp)_z \rho^q(x, \mathbf{b}_\perp, \mathbf{k}_\perp; \mathcal{W}). \quad (160)$$

These quantities are both dependent on the path of the gauge-kink, since the Wigner distribution defined by

$$\begin{aligned} \rho^q(x, \mathbf{k}_\perp, \mathbf{b}_\perp; \mathcal{W}) &= \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{-i \Delta_\perp \cdot \mathbf{b}_\perp} \frac{1}{2} \int \frac{d\xi^- d^2 \xi_\perp}{(2\pi)^3} e^{i(x \bar{P}^+ \xi^- - \mathbf{k}_\perp \cdot \xi_\perp} \\ &\times \langle P'^+, \frac{\Delta_\perp}{2}, S | \bar{\psi}(0) \gamma^+ \mathcal{W} \psi(\xi) | P^+, -\frac{\Delta_\perp}{2}, S \rangle |_{\xi=0}, \end{aligned} \quad (161)$$

is generally dependent on the path connecting the two space-time points ξ and 0. Physically interesting paths are the following three. The first is the future-pointing light-like staple path $\mathcal{W}_{0\xi}^{+LC}$ (see Fig.3(a)) corresponding to the kinematics of semi-inclusive hadron productions. As seen from Fig.3(a), by using the straightline path $\mathcal{W}_{\xi^-, \xi_\perp, \xi'^-, \xi'_\perp}^{(sl)}$ connecting the two space-time points by a straight line, it is represented as $\mathcal{W}_{0\xi}^{+LC} \equiv \mathcal{W}_{0^-, \mathbf{0}_\perp, \infty^-, \mathbf{0}_\perp}^{(sl)} \mathcal{W}_{\infty^-, \mathbf{0}_\perp, \infty^-, \xi_\perp}^{(sl)} \mathcal{W}_{\infty^-, \xi_\perp, \xi^-, \xi_\perp}^{(sl)}$. The second is the past-pointing light-like staple path $\mathcal{W}_{0\xi}^{-LC}$ corresponding to the kinematics of Drell-Yan processes. It is represented as $\mathcal{W}_{0\xi}^{-LC} \equiv \mathcal{W}_{0^-, \mathbf{0}_\perp, -\infty^-, \mathbf{0}_\perp}^{(sl)} \mathcal{W}_{-\infty^-, \mathbf{0}_\perp, -\infty^-, \xi_\perp}^{(sl)} \mathcal{W}_{-\infty^-, \xi_\perp, \xi^-, \xi_\perp}^{(sl)}$, as illustrated in Fig.3(b). The last is the straightline path connecting the two space-time points $(0^-, \mathbf{0}_\perp)$ and (ξ^-, ξ_\perp) directly, which gives $\mathcal{W}^{straight} \equiv \mathcal{W}_{0^-, \mathbf{0}_\perp, \xi^-, \xi_\perp}^{(sl)}$.

Burkardt primarily concentrated on the difference between the two average quark transverse momenta and also the difference between the two OAMs, corresponding to the two gauge-link paths, i.e. the future-pointing light-like staple path \mathcal{W}^{+LC} and the straightline path $\mathcal{W}^{straight}$. When evaluating the average transverse momentum $\langle k_\perp^i \rangle_{+LC} \equiv \int dx d^2 b_\perp d^2 k_\perp k_\perp^i \rho^q(x, \mathbf{b}_\perp, \mathbf{k}_\perp, \mathcal{W}^{+LC})$, the factor k_\perp^i can be translated into a derivative $-i \frac{\partial}{\partial \xi_\perp^i}$ acting on the operator $\bar{\psi}(0) \gamma^+ \mathcal{W}_{0\xi}^{+LC} \psi(\xi)$, whose matrix element is subsequently evaluated for $\xi_\perp = \mathbf{0}_\perp$. With use of the familiar derivative formula on the Wilson line,¹¹¹

$$\begin{aligned} -i \frac{\partial}{\partial \xi_\perp^i} \mathcal{W}_{0\xi}^{+LC} \Big|_{\xi=0} &= -\mathcal{W}_{0-\mathbf{0}_\perp, \infty-\mathbf{0}_\perp} A_\perp^i(\infty^-, \mathbf{0}_\perp) \mathcal{W}_{\infty-\mathbf{0}_\perp, 0-\mathbf{0}_\perp} \\ &+ \int_{0-}^{\infty} dz^- \mathcal{W}_{0-\mathbf{0}_\perp, z^--\mathbf{0}_\perp} \partial^i A^+(z^-, \mathbf{0}_\perp) \mathcal{W}_{z^--\mathbf{0}_\perp, 0-\mathbf{0}_\perp}, \end{aligned} \quad (162)$$

one readily obtains

$$\begin{aligned} \langle k_\perp^i \rangle_{+LC} &= \mathcal{N} \int d^3 r \langle PS | \bar{\psi}(\mathbf{r}) \gamma^+ \left(\frac{1}{i} \nabla_\perp^i - g A_\perp^i(r^-, \mathbf{r}_\perp) \right. \\ &\quad \left. - \int_{r^-}^{\infty} dz^- \mathcal{W}_{r^--\mathbf{r}_\perp, z^--\mathbf{r}_\perp} F^{+i}(z^-, \mathbf{r}_\perp) \mathcal{W}_{z^--\mathbf{r}_\perp, r^--\mathbf{r}_\perp} \right) \psi(\mathbf{r}) | PS \rangle, \end{aligned} \quad (163)$$

with $\mathcal{N} = 1 / \langle PS | PS \rangle$. Similarly, it can be shown that the average transverse momentum corresponding to the straightline path $\mathcal{W}^{straight}$ is given by

$$\begin{aligned} \langle k_\perp^i \rangle^{straight} &= \mathcal{N} \int d^3 r \langle PS | \bar{\psi}(\mathbf{r}) \gamma^+ \left(\frac{1}{i} \nabla_\perp^i - g A_\perp^i(r^-, \mathbf{r}_\perp) \right) \psi(\mathbf{r}) | PS \rangle. \end{aligned} \quad (164)$$

The difference between these two quantities is therefore given by

$$\begin{aligned} \langle k_\perp^i \rangle_{+LC} - \langle k_\perp^i \rangle^{straight} &= -\mathcal{N} \int d^3 r \langle PS | \bar{\psi}(\mathbf{r}) \gamma^+ \\ &\quad \times \int_{r^-}^{\infty} dz^- \mathcal{W}_{r^--\mathbf{r}_\perp, z^--\mathbf{r}_\perp} F^{+i}(z^-, \mathbf{r}_\perp) \mathcal{W}_{z^--\mathbf{r}_\perp, r^--\mathbf{r}_\perp} \psi(\mathbf{r}) | PS \rangle, \end{aligned} \quad (165)$$

although one can show that $\langle k_\perp^i \rangle^{straight}$ vanishes by time-reversal invariance.

As pointed out by Burkardt, the quantity on the r.h.s. is nothing but the well-known Qiu-Sterman matrix element.¹¹² According to him,⁶² this quantity can be interpreted as the change of transverse momentum for the struck quark as it leaves the target after being struck by the virtual photon in the semi-inclusive DIS processes. The legitimacy of this interpretation can most easily be seen by taking the light-cone gauge. In this gauge, the Wilson lines along the light-like direction become unity and the relevant component of the field-strength tensor, for example, F^{+y} , reduces to

$$\begin{aligned} -\sqrt{2} g F^{+y} &= -g F^{0y} - g F^{zy} = g(E^y - B^x) \\ &= g [\mathbf{E} + (\mathbf{v} \times \mathbf{B})]^y, \end{aligned} \quad (166)$$

which represents the y -component of the color Lorentz force acting on a particle that moves with the velocity of light in the $-z$ direction, i.e. $\mathbf{v} = (0, 0, -1)$, that is the direction of the momentum transfer in the semi-inclusive DIS reactions. This motivates him the semiclassical interpretation of the matrix element of (165) as the average transverse momentum of the ejected quark generated by the average color-Lorentz force from the spectator as it leaves the target.

A similar analysis can be carried out also for the quark OAMs, although it needs an extra care. That is, when one evaluates $\langle L_z^q \rangle^{+LC} = \int dx d^2b_\perp d^2k_\perp (\mathbf{b}_\perp \times \mathbf{k}_\perp)^z \rho^q(x, \mathbf{b}_\perp, \mathbf{k}_\perp; \mathcal{W}^{+LC})$, the factor \mathbf{b}_\perp can be translated into a derivative $-i \frac{\partial}{\partial \Delta_\perp^+}$ acting on the matrix element $\langle P' S' | \bar{\psi}(0) \gamma^+ \mathcal{W}_{0\xi}^{+LC} \psi(\mathbf{r}) | PS \rangle = \langle \bar{P} - \frac{\Delta}{2}, S' | \bar{\psi}(0) \gamma^+ \mathcal{W}_{0\xi}^{+LC} \psi(\mathbf{r}) | \bar{P} + \frac{\Delta}{2}, S \rangle$ contained in the definition of the Wigner distribution $\rho^q(x, \mathbf{b}_\perp, \mathbf{k}_\perp; \mathcal{W}^{+LC})$. Hatta carried out this nontrivial operation by making use of a parametrization of the above nucleon matrix element in terms of the twist-3 quark-gluon correlation.^{57, 58} Using the notation of Burkardt,⁶² this gives

$$\begin{aligned} & \langle L_z^q \rangle^{+LC} \\ &= \mathcal{N} \int d^3r \langle PS | \bar{\psi}(\mathbf{r}) \gamma^+ \left\{ \left[\mathbf{r} \times \left(\frac{1}{i} \nabla - g \mathbf{A} \right) \right]^z - \int_{r^-}^{\infty} dz^- \mathcal{W}_{r^-, \mathbf{r}_\perp, z^-, \mathbf{r}_\perp} \right. \\ & \quad \left. \times g (x F^{+y}(z^-, \mathbf{r}_\perp) - y F^{+x}(z^-, \mathbf{r}_\perp)) \mathcal{W}_{r^-, \mathbf{r}_\perp, r^-, \mathbf{r}_\perp} \right\} \psi(\mathbf{r}) | PS \rangle. \end{aligned} \quad (167)$$

Similarly, as first noticed by Ji, Kiong, and Yuan,⁷⁵ the OAM corresponding to the straightline path $\mathcal{W}_{straight}$ is given by

$$\langle L_z^q \rangle^{straight} = \mathcal{N} \int d^3r \langle PS | \bar{\psi}(\mathbf{r}) \gamma^+ \left[\mathbf{r} \times \left(\frac{1}{i} \nabla - g \mathbf{A} \right) \right]^z \psi(\mathbf{r}) | PS \rangle. \quad (168)$$

One therefore finds for the difference

$$\begin{aligned} & \langle L_z^q \rangle^{+LC} - \langle L_z^q \rangle^{straight} = -\mathcal{N} \int d^3r \langle PS | \bar{\psi}(\mathbf{r}) \gamma^+ \int_{r^-}^{\infty} dz^- \mathcal{W}_{r^-, \mathbf{r}_\perp, z^-, \mathbf{r}_\perp} \\ & \quad \times g (x F^{+y}(z^-, \mathbf{r}_\perp) - y F^{+x}(z^-, \mathbf{r}_\perp)) \mathcal{W}_{z^-, \mathbf{r}_\perp, r^-, \mathbf{r}_\perp} \psi(\mathbf{r}) | PS \rangle. \end{aligned} \quad (169)$$

Analogous to the previous semiclassical interpretation of $-g F^{+i}(r^-, \mathbf{r}_\perp)$ as the transverse force acting on the active quark along its trajectory, Burkardt gave an interpretation that

$$T^z(r^-, \mathbf{r}_\perp) \equiv -g (x F^{+y}(r^-, \mathbf{r}_\perp) - y F^{+x}(r^-, \mathbf{r}_\perp)), \quad (170)$$

represents the z -component of the *torque* that acts on a particle moving with the velocity of light in the $-z$ direction - the direction in which the ejected quark moves. Consequently, the difference between the two OAMs, i.e. $\langle L_z^q \rangle^{+LC}$ and $\langle L_z^q \rangle^{straight}$ is interpreted as the change of the OAM as the quark moves through the color field created by the spectators. We point out that these two OAMs $\langle L_z^q \rangle^{+LC}$ and $\langle L_z^q \rangle^{straight}$ are nothing but the “canonical” and “mechanical” OAMs, L_{can} and L_{mech} , respectively.

The analysis of Burkardt is limited only to the difference between the two average transverse momenta and the difference between the two OAMs. We find it very

interesting to reconsider his analysis from a different viewpoint. Let us start with the following relations :

$$\begin{aligned} \langle k_{\perp}^i \rangle^{\pm LC} &= \mathcal{N} \int d^3r \langle PS | \bar{\psi}(\mathbf{r}) \gamma^+ \left(\frac{1}{i} \nabla_{\perp}^i - g A_{\perp}^i(r^-, \mathbf{r}_{\perp}) \right. \\ &\quad \left. - \int_{r^-}^{\pm\infty} dz^- \mathcal{W}_{r^-, \mathbf{r}_{\perp}, z^-, \mathbf{r}_{\perp}} F^{+i}(z^-, \mathbf{r}_{\perp}) \mathcal{W}_{z^-, \mathbf{r}_{\perp}, r^-, \mathbf{r}_{\perp}} \right) \psi(\mathbf{r}) | PS \rangle. \end{aligned} \quad (171)$$

and

$$\begin{aligned} \langle L_z^q \rangle^{\pm LC} &= \mathcal{N} \int d^3r \langle PS | \bar{\psi}(\mathbf{r}) \gamma^+ \left\{ \left[\mathbf{r} \times \left(\frac{1}{i} \nabla - g \mathbf{A} \right) \right]^z - \int_{r^-}^{\pm\infty} \mathcal{W}_{r^-, \mathbf{r}_{\perp}, z^-, \mathbf{r}_{\perp}} \right. \\ &\quad \left. \times g (x F^{+y}(z^-, \mathbf{r}_{\perp}) - y F^{+x}(z^-, \mathbf{r}_{\perp})) \mathcal{W}_{z^-, \mathbf{r}_{\perp}, r^-, \mathbf{r}_{\perp}} \right\} \psi(\mathbf{r}) | PS \rangle. \end{aligned} \quad (172)$$

Here, we have given the average transverse momentum and the OAM not only for the future-pointing light-like staple path but also for the past-pointing one. As pointed out by Hatta, the two OAMs $\langle L_z^q \rangle^{\pm LC}$ are actually shown to coincide due to parity and time-reversal (PT) symmetry, and they can be identified with the (gauge-invariant) canonical OAM as

$$\langle L_z^q \rangle^{+LC} = \langle L_z^q \rangle^{-LC} = \frac{1}{2} (\langle L_z^q \rangle^{+LC} + \langle L_z^q \rangle^{-LC}) = L_{can}. \quad (173)$$

In fact, the quantity appearing in the 2nd term of $\langle L_z^q \rangle^{\pm LC}$ can be written as

$$\begin{aligned} &- \int_{r^-}^{\pm\infty} dz^- \mathcal{W}_{r^-, \mathbf{r}_{\perp}, z^-, \mathbf{r}_{\perp}} F^{+\mu}(z^-, \mathbf{r}_{\perp}) \mathcal{W}_{z^-, \mathbf{r}_{\perp}, r^-, \mathbf{r}_{\perp}} \\ &= - \int dz^- \kappa(z^- - r^-), \mathcal{W}_{r^-, \mathbf{r}_{\perp}, z^-, \mathbf{r}_{\perp}} F^{+\mu}(z^-, \mathbf{r}_{\perp}) \mathcal{W}_{z^-, \mathbf{r}_{\perp}, r^-, \mathbf{r}_{\perp}}, \end{aligned} \quad (174)$$

with use of the functions

$$\kappa(z^-) = \pm \theta(\pm z^-), \quad (175)$$

depending on the two choices of path. The above quantity precisely coincides with the physical component of the gluon defined by Hatta,⁵⁶

$$\begin{aligned} &A_{phys}^{\mu}(r^-, \mathbf{r}_{\perp}) \\ &= - \int dz^- \kappa(z^- - r^-) \mathcal{W}_{r^-, \mathbf{r}_{\perp}, z^-, \mathbf{r}_{\perp}} F^{+\mu}(z^-, \mathbf{r}_{\perp}) \mathcal{W}_{z^-, \mathbf{r}_{\perp}, r^-, \mathbf{r}_{\perp}}. \end{aligned} \quad (176)$$

Plugging this into (172), one thus obtains

$$\begin{aligned} \langle L_z^q \rangle^{\pm LC} &= \mathcal{N} \int d^3r \langle PS | \bar{\psi}(\mathbf{r}) \gamma^+ \left\{ \left[\mathbf{r} \times \left(\frac{1}{i} \nabla - g \mathbf{A}(r^-, \mathbf{r}_{\perp}) \right) \right]^z \right. \\ &\quad \left. + (\mathbf{r} \times \mathbf{A}_{phys}(r^-, \mathbf{r}_{\perp}))^z \right\} \psi(\mathbf{r}) | PS \rangle \\ &= \mathcal{N} \int d^3r \langle PS | \bar{\psi}(\mathbf{r}) \gamma^+ \left[\mathbf{r} \times \left(\frac{1}{i} \nabla - g \mathbf{A}_{pure}(r^-, \mathbf{r}_{\perp}) \right) \right]^z \psi(\mathbf{r}) | PS \rangle. \\ &= L_{can}. \end{aligned} \quad (177)$$

The r.h.s. of this equation in fact reproduces the theoretical expression for the gauge-invariant canonical momentum L_{can} . What is important here is that, because of the equality (173), the definition of the canonical OAM is independent of the two choice of the light-like paths relevant to the two physical DIS processes.

As we shall see below, however, this is not the case for the average transverse momenta. In fact, exactly in the same way as the manipulation above, one can show that

$$\langle k_{\perp}^i \rangle^{\pm LC} = \mathcal{N} \int d^3r \langle PS | \bar{\psi}(\mathbf{r}) \left[\gamma^+ \left(\frac{1}{i} \nabla_{\perp}^i - g A^i(r^-, \mathbf{r}_{\perp}) \right) + g A_{phys}^i(r^-, \mathbf{r}_{\perp}) \right] \psi(\mathbf{r}) | PS \rangle. \quad (178)$$

This therefore gives

$$\begin{aligned} \langle k_{\perp}^i \rangle^{\pm LC} \\ = \mathcal{N} \int d^3r \langle PS | \bar{\psi}(\mathbf{r}) \gamma^+ \left(\frac{1}{i} \nabla_{\perp}^i - g A_{pure}^i(r^-, \mathbf{r}_{\perp}) \right) \psi(\mathbf{r}) | PS \rangle. \end{aligned} \quad (179)$$

Formally, the r.h.s. of this relation is the defining equation of canonical transverse momentum $\langle k_{\perp}^i \rangle_{can}$. A problem here is that the two average transverse momenta $\langle k_{\perp}^i \rangle^{\pm LC}$ do not agree with each other. In fact, from PT symmetry, they have opposite sign with equal magnitude¹¹³

$$\langle k_{\perp}^i \rangle^{-LC} = - \langle k_{\perp}^i \rangle^{+LC}. \quad (180)$$

This means that the definition of the average transverse canonical momentum is not universal. Putting it in another way, while the potential angular momentum L'_{pot} defined by

$$L'_{pot} \equiv L_{mech} - L_{can}, \quad (181)$$

may basically be a universal quantity, the potential momentum defined by

$$\langle k_{\perp}^i \rangle_{pot} \equiv \langle k_{\perp}^i \rangle_{mech} - \langle k_{\perp}^i \rangle_{can} \quad (182)$$

is not a path-independent quantity. Since $\langle k_{\perp}^i \rangle_{mech} = 0$ by time-reversal symmetry, one can also say that the potential momentum has just opposite sign with equal magnitude for semi-inclusive and Drell-Yan processes. One natural possibility of defining the canonical transverse momentum might be to take an average, i.e.

$$\langle k_{\perp}^i \rangle_{can} \equiv \frac{1}{2} (\langle k_{\perp}^i \rangle^{+LC} + \langle k_{\perp}^i \rangle^{-LC}), \quad (183)$$

which gives

$$\langle k_{\perp}^i \rangle_{can} = 0. \quad (184)$$

It however does not correspond to a single DIS process, thereby making its physical meaning obscure. This consideration in turn indicates somewhat peculiar nature of the canonical momentum corresponding to the transverse motion.

What is curious here is the origin of the difference between the case of the average transverse momentum and that of the OAM. A plausible reason might be the

following. In the argument of OAM, we are considering its longitudinal component, i.e. the component along the direction of nucleon momentum. On the other hand, in the case of average transverse momentum, we are dealing with the components perpendicular to the direction of nucleon momentum. To understand the significance of this difference, we recall here more familiar discussion on the longitudinal momentum fractions of quarks and gluons. It starts with the standard gauge-invariant decomposition of the QCD energy momentum tensor given as follows :

$$T^{\mu\nu} = T_q^{\mu\nu} + T_G^{\mu\nu}, \quad (185)$$

with

$$T_q^{\mu\nu} = \frac{1}{2} \bar{\psi} (\gamma^\mu i D^\nu + \gamma^\nu i D^\mu) \psi, \quad (186)$$

$$T_G^{\mu\nu} = 2 \text{Tr} [F^{\mu\alpha} F_\alpha^\nu] + \frac{1}{2} \text{Tr} F^2. \quad (187)$$

The quark part of the above QCD energy-momentum tensor is the famous Belinfante symmetric tensor, or the mechanical energy-momentum tensor of quarks. Eq.(185) therefore gives the mechanical decomposition of the QCD energy momentum tensor.

On the other hand, the “seemingly” covariant version of the canonical decomposition, takes the following form :

$$T^{\mu\nu} = T_q'^{\mu\nu} + T_G'^{\mu\nu}, \quad (188)$$

with

$$T_q'^{\mu\nu} = \frac{1}{2} \bar{\psi} (\gamma^\mu i D_{pure}^\nu + \gamma^\nu i D_{pure}^\mu) \psi, \quad (189)$$

$$T_G'^{\mu\nu} = -\text{Tr} [F^{\mu\alpha} D_{pure}^\nu A_{\alpha,phys} + F^{\nu\alpha} D_{pure}^\mu A_{\alpha,phys}] + \frac{1}{2} \text{Tr} F^2. \quad (190)$$

Note that the quark part of this decomposition stands for the canonical energy-momentum tensor, aside from unphysical gauge degrees of freedom. Eq.(188) therefore gives the canonical decomposition of the QCD energy momentum tensor. What do these two different decompositions predict for the momentum sum rule of QCD ? Utilizing the freedom of gauge choice, one can take the light-cone gauge ($A^+ = 0$). In this case, we can set

$$A_{phys}^+ \rightarrow 0, \quad A_{pure}^+ \rightarrow 0, \quad (191)$$

$$D^+ \equiv \partial^+ - i g A^+ \rightarrow \partial^+, \quad D_{pure}^+ \equiv \partial^+ - i g A_{pure}^+ \rightarrow \partial^+, \quad (192)$$

$$F^{+\alpha} = \partial^+ A^\alpha - \partial^\alpha A^+ - g [A^+, A^\alpha] \rightarrow \partial^+ A^\alpha. \quad (193)$$

Consequently, T^{++} component in *either* of the above two decompositions reduce to the following simple form,

$$T^{++} = i \psi_+^\dagger \partial^+ \psi_+ + \text{Tr} (\partial^+ \mathbf{A}_\perp)^2, \quad (194)$$

where $\psi^+ \equiv \frac{1}{2} \gamma^+ \gamma^- \psi$ with $\gamma^\pm = (\gamma^0 \pm \gamma^3) / \sqrt{2}$ is the so-called good component of the quark field. As emphasized by Jaffe many years ago,¹¹⁴ interaction-dependent

part drops in the light-cone gauge and infinite-momentum frame. Thus, from

$$\langle P_\infty | T^{++} | P_\infty \rangle / 2 (P_\infty^+)^2 = 1, \quad (195)$$

we are led to the standard momentum sum rule of QCD given as

$$\langle x \rangle^q + \langle x \rangle^G = 1. \quad (196)$$

Note that even the canonical decomposition gives this standard sum rule, contrary to the claim in the original paper by Chen et al.²⁹ The point is that the difference between the canonical energy momentum tensor

$$T_q'^{++} = \frac{1}{2} \bar{\psi} (\gamma^+ i \partial^+ + \gamma^+ i \partial^+) \psi, \quad (197)$$

and the mechanical energy momentum tensor

$$T_q^{++} = \frac{1}{2} \bar{\psi} (\gamma^+ i D^+ + \gamma^+ i D^+) \psi, \quad (198)$$

does not have effect on the longitudinal momentum sum rule after taking the light-cone gauge ($A^+ = 0$). We have seen that this simple argument does not hold for the average transverse momentum, and that there is no universal definition of the canonical momentum in this case. It seems to us that this once again shed light on “unphysical” (or “mathematical”) nature of the idea of canonical momentum, at least in its most general context. By some deep reason^j, such a discrepancy does not occur for the longitudinal component of the OAMs. Nevertheless, by drawing on all the arguments in this sections, we feel that what has closer relationship with physical observables is the mechanical OAM rather than the canonical OAM.

After overiewing various aspects of the gauge-invariant nucleon spin decomposition problem, we think it useful to revisit the consideration on the Stueckelberg symmetry given in sect.3. According to Lorcé,^{51–55} the Stueckelberg symmetry dictates existence of infinitely many decomposition of the gluon field into its physical and pure-gauge component, and this in turn leads to infinitely many GIEs of Jaffe-Manohar decomposition of the nucleon spin. However, summing up the consideration so far, it appears that the gauge symmetry plays only the secondary role in the existence of plural forms of nucleon spin decomposition. First, we now understand that there are in principle infinitely many definitions of relativistic spin operator (we are supposing here, for example, the existence of many definitions of transverse spin), the origin of which can be attributed to the relativity not the gauge symmetry.^k Next, suppose that we are considering one of these spin operators, and that it does not have manifest gauge-invariance. Such an operator can readily (or trivially) be made gauge-invariant by making use of Wilson lines. This

^jWe conjecture that it is not unrelated to the fact that only the component of the gauge field along the propagation direction can be decomposed into the physical and pure-gauge parts in a gauge- and frame-independent way.

^kAs already pointed out before, the ultimate origin of it can be traced back to the fact that successive operations of Lorentz boosts generate spin rotation.

is the point at which the path-dependence arises in the definition of the spin operators. Nevertheless, from the context of physical application, there are actually restricted numbers of paths, at least at the dominant order of twist expansion. (We are supposing here, for instance, the future-pointing and past-pointing light-like staple paths, which appears in the definitions of TMDs and/or Wigner distributions.) According to Lorcé's viewpoint, they might be called different GIEs. However, we do not necessarily need to use such a conceptually strange notion like the GIEs. In fact, plainer interpretation would be that there are simply two different definitions of relativistic spin observables (or quasi observables), both of which correspond to different experimental settings. Despite these general statements, exceptional features of the longitudinal nucleon spin sum rule should not be forgotten. First, the relativity does not interude a unique definition of the sum rule, essentially because the helicity of the massless gluon is a Lorentz-invariant concept. Second, there is no essential path-dependence in the definition of the gluon spin operator, which can be defined as the 1st moment of a collinear distribution function. (The relevant path here is just a light-like straightline path.) This enables us to get gauge- as well as Lorentz-frame-independent decompositions of the longitudinal nucleon spin in a traditional sense.

8. Lattice QCD Studies of Nucleon Spin Contents

As pointed out in Sect.1, among the 4 pieces of the longitudinal nucleon spin decompositions, only the intrinsic quark spin contribution has been fairly precisely determined through experiments. Empirical information on the other parts is still very poor. Fortunately, there have been a great progress from the theoretical side. That is, we can now get valuable information from the lattice QCD simulations, which provides us with a powerful tool for handling nonperturbative QCD. Over the last few years, the two lattice QCD collaborations carried out extensive studies on the nucleon spin contents, although within the so-called quenched approximation.^{115–119} The basis of these analyses is the well-known Ji sum rule :

$$J^q = \frac{1}{2} [A_{20}^q(0) + B_{20}^q(0)], \quad (199)$$

$$J^G = \frac{1}{2} [A_{20}^G(0) + B_{20}^G(0)], \quad (200)$$

with $A_{20}^q(0)$ and $B_{20}^q(0)$ being the forward limits of the so-called generalized form factors, which are related to the forward limits of the unpolarized GPDs as

$$A_{20}^q(0) = \int_{-1}^1 x H^q(x, 0, 0) dx, \quad (201)$$

$$B_{20}^q(0) = \int_{-1}^1 x E^q(x, 0, 0) dx, \quad (202)$$

and similarly for the gluon part. Here, we confine to the quark part, since the study of the gluon part is still a difficult challenge even for the lattice QCD. The lattice

QCD simulations concentrates on evaluating the four quantities $A_{20}^{u\pm d}(0)$, $B_{20}^{u\pm d}(0)$, which is necessary to get separate knowledge on J^u and J^d . Once the total angular momentum J^q of the quark with a particular flavor is known, the quark OAM is obtained through the relation,

$$L^q = J^q - \frac{1}{2} \Delta\Sigma^q. \quad (203)$$

Here, $\Delta\Sigma^q$ is the 1st moment of the familiar longitudinally polarized quark distribution,

$$\Delta\Sigma^q = \int_{-1}^1 \Delta q(x) dx, \quad (204)$$

with the corresponding flavor q . Needless to say, the quark OAM obtained in this way corresponds to the “mechanical” OAM not the “canonical” OAM.

Shown in Table 2 are the lattice QCD predictions for $2J \equiv 2(J^u + J^d + J^s)$, $\Delta\Sigma \equiv \Delta u + \Delta d + \Delta s$, and $2L \equiv 2(L^u + L^d + L^s)$ by the LHPC^{115–117} and QCDSF-UKQCD groups.^{118,119} For the sake of comparison, we also show here the corresponding predictions of the chiral quark soliton model (CQSM) evolved to the energy scale of $Q^2 = 4 \text{ GeV}^2$,^{120,121} which corresponds to the renormalization scale of lattice QCD calculations. The prediction of the CQSM is shown, because it is a particularly successful model of the nucleon structure functions.^{122–131} In particular, it is almost only one effective model of the nucleon, which is able to explain the observed smallness of the quark spin fraction of the nucleon without any fine-tuning. Moreover, this unique prediction of the model is inseparably connected with its basic physical picture of the nucleon as a rotating hedgehog, which in turn predicts fairly large orbital angular momentum of quarks.^{132,133} As seen from the table, this interesting prediction of the CQSM does not seem to be supported by the lattice QCD predictions of the LHPC and the QCDSF-UKQCD Collaborations. The results of both groups show that both the u - and d -quark OAMs carry sizable amount (nearly 20 %) of the nucleon spin. However, their contributions to the net nucleon spin tend to cancel in such a way that

$$2L^u \simeq -0.2, \quad 2L^d \simeq +0.2, \quad 2L^{u+d} \simeq 0. \quad (205)$$

Table 2. Lattice QCD predictions for the nucleon spin contents by the LHPC^{115–117} and QCDSF-UKQCD Collaborations.^{118,119} Also shown for comparison are the predictions of the chiral quark soliton model evolved to the renormalization scale of lattice QCD.

	LHPC	QCDSF-UKQCD	CQSM ($Q^2 = 4 \text{ GeV}^2$)
$2J$	0.426(48)	0.452(26)	0.676
$\Delta\Sigma$	0.409(34)	0.402(48)	0.318
$2L$	0.005(52)	0.050(54)	0.358

Naturally, one must be careful about large uncertainties inherent in the lattice QCD calculations at this stage. They suffer from various limitations, which come from the quenched approximation, the finite-size effects of the lattice, large pion mass effects and/or the ambiguities in the chiral extrapolation procedures, etc. Also noteworthy is the fact that, in the simulation by the LHPC and the QCDSF-UKQCD groups, only the contributions of connected-insertion (CI) were taken into account and those of the disconnected-insertion (DI) were totally left out.

More recently, χ QCD Collaboration carried out a challenging study of nucleon spin contents by including the DI contributions as well and found that they in fact have sizable effects.^{134, 135} Their results are shown in Table 3. They confirmed the results by the LHPC and the QCDSF-UKQCD collaborations that the CI contribution to the net quark OAM is certainly very small. However, they found that the DI contribution to the same quantity is very large. As a consequence, their result shows that nearly half of the nucleon spin comes from the quark OAM (“mechanical” OAM). This number is even larger than the prediction of the CQSM, although they are consistent in a qualitative sense. In view of the previously-mentioned various uncertainties of the lattice QCD calculation, it would be premature to draw a decisive conclusion at the present stage. Nevertheless, their analysis clearly reminds us of the fact that some of the nucleon observables are very sensitive to the introduction of the DI contributions, which are thought to simulate the pion clouds effects dictated by the spontaneous chiral symmetry breaking of QCD vacuum. This means that, in order to get realistic predictions for internal structures of the nucleon in the framework of lattice QCD, more serious account of the DI contributions is absolutely necessary. Also highly desirable is to carry out calculations with dynamical fermions.

Table 3. Lattice QCD estimate of the contributions of connected insertions (CI) and the disconnected insertions (DI) to the nucleon spin contents by the χ QCD Collaboration.^{134, 135}

	CI ($u + d$)	DI ($u + d + s$)	sum
$2J$	0.629(51)	0.092(14)	0.72(8)
$\Delta\Sigma$	0.62(9)	- 0.36(3)	0.25(12)
$2L$	0.01(10)	0.46(3)	0.47(13)

So far, our eyes are mainly turned on flavor singlet combination (or the net contribution) of the quark OAMs. Also very interesting is the isovector combination, i.e. the difference of OAMs carried by u -quark and d -quark. It should be emphasized that the lattice QCD predictions for this flavor-nonsinglet quantity are expected to be quantitatively more trustable than that for the flavor-singlet quantity L^{u+d} , because there is no DI contribution to the former, which is harder to estimate reliably. As already pointed out, the predictions of the LHPC and QCDSF-UKQCD groups are that L^u is negative and L^d is positive, which leads to a remarkable

prediction that $L^u - L^d$ is sizably negative

$$2 L^{u-d} \simeq -0.4. \quad (206)$$

This must be a surprise. In fact, it sharply contradicts the prediction of the familiar quark model like the MIT bag model. To explain it, we first recall the nucleon spin sum rule obtained within the familiar MIT bag model in both of the isoscalar and isovector channels. They are given by

$$2 J^{u+d} = \Delta \Sigma^{u+d} + 2 L^{u+d}, \quad (207)$$

$$2 J^{u-d} = \Delta \Sigma^{u-d} + 2 L^{u-d}, \quad (208)$$

where $2 J^{u+d} = 1$, $2 J^{u-d} = 5/3$, while

$$\Delta \Sigma^{u+d} = \int_0^R \left\{ [f(r)]^2 - \frac{1}{3} [g(r)]^2 \right\} r^2 dr, \quad (209)$$

$$\Delta \Sigma^{u-d} = \frac{5}{3} \Delta \Sigma^{u+d}, \quad (210)$$

and

$$L^{u+d} = \frac{2}{3} \int_0^R [g(r)]^2 r^2 dr, \quad (211)$$

$$L^{u-d} = \frac{5}{3} L^{u+d}. \quad (212)$$

Here, R is the bag radius, while $f(r)$ and $g(r)$ are the radial wave functions of the upper and lower components of the ground state of the MIT bag model, given in the form :

$$\psi_{g.s.}(\mathbf{r}) = \begin{pmatrix} f(r) \chi_s \\ i g(r) \boldsymbol{\sigma} \cdot \hat{\mathbf{r}} \chi_s \end{pmatrix}, \quad (213)$$

with the normalization

$$\int_0^R \{ [f(r)]^2 + [g(r)]^2 \} r^2 dr = 1. \quad (214)$$

Note that the limiting case of non-relativistic quark model is obtained by setting the lower component to be zero, i.e. $g(r) \equiv 0$. In this limit, we have

$$\Delta \Sigma^{u+d} = 1, \quad L^{u+d} = 0, \quad (215)$$

and

$$\Delta \Sigma^{u-d} = \frac{5}{3}, \quad L^{u-d} = 0. \quad (216)$$

This just reconfirms the fact that, in the MIT bag model, the quark OAMs come from the lower P -wave component of the ground-state wave function. For a typical bag parameter, which gives $\Delta \Sigma^{u+d} \simeq 0.7$, one would find that

$$2 L^{u+d} \simeq 0.30, \quad 2 L^{u-d} \simeq 0.50, \quad (217)$$

which especially means that $L^u - L^d$ is positive with sizable magnitude. This is in sharp contradiction to the afore-mentioned prediction of the lattice QCD, $2L^{u-d} \simeq -0.4$. We recall that the discrepancy between the EMC observation $\Delta\Sigma \simeq (0.2 - 0.3)$ and the predictions of the standard quark models (remember for example, the prediction of the naive quark model $\Delta\Sigma^{u+d} = 1$, or the prediction of the MIT bag model $\Delta\Sigma^{u+d} \simeq 0.7$) was called the “nucleon spin crisis”. Now, the discrepancy between the isovector combination of the quark OAMs pointed out above seems more drastic, in the sense that even their signs are different. One might then call it “another” nucleon spin crisis.

There can be two possible origins of this discrepancy. The first is the possibility of significant numerical difference between the two OAMs, i.e. the canonical and mechanical OAMs. The OAM difference $L^u - L^d$ calculated above in the MIT bag model roughly corresponds to the canonical OAM, although the MIT bag model is not a gauge theory. On the other hand, the OAM $L^u - L^d$ extracted from the lattice QCD analysis with help of the Ji sum rule is the mechanical OAM. There is no reason to expect that their difference is small.

The second possible origin is the strong scale dependence of $L^u - L^d$, which we shall discuss below. Remember first that the renormalization scale of the lattice QCD calculation corresponds to relatively high-energy scale as $Q^2 \simeq 4 \text{ GeV}^2$. On the other hand, the energy scale of low energy models like the MIT bag model is believed to be much lower, say $Q^2 \simeq (0.4 \sim 0.6 \text{ GeV})^2$. This makes no big difference in the case of flavor singlet quark spin, because it is nearly a scale-independent quantity, except in the extremely low energy regions where the framework of the perturbative renormalization group becomes untrusted.¹ However, the isovector combination of the quark OAMs turns out to be strongly scale-dependent quantities.^{120, 121, 136} Thomas then claims that this strong scale-dependence of $L^u - L^d$ is likely to resolve the above-mentioned discrepancy at least partially.^{136, 137} This explanation was criticized by Wakamatsu,¹³⁸ however. Thomas’ analysis starts from an estimate of the u - and d -quark OAMs based on the improved cloudy bag model which also takes account of the exchange current contribution associated with the one-gluon-exchange hyperfine interactions. Those model predictions are regarded as initial scale values corresponding to a very low energy scale, say, 0.4 GeV . Then, by solving the QCD evolution equation for the u - and d -quark OAMs first derived by Ji, Tang, and Hoodbhoy,¹³⁹ he found that the OAMs of u - and d -quarks cross over around the scale of 0.5 GeV . This crossover of L^u and L^d is just what is required from the consistency with the lattice QCD results given at the scale of $Q^2 \simeq 4 \text{ GeV}^2$. (Actually, a careful observation reveals that the discrepancy between Thomas’ prediction and the lattice QCD predictions at the scale $Q^2 = 4 \text{ GeV}^2$ is fairly large.) As pointed out in Ref.138, however, the starting energy of evolution used in his analysis is fairly low, and, at such low energy scales, L^{u-d} shows tremendously strong scale depen-

¹Here, we are supposing the gauge-invariant $\overline{\text{MS}}$ factorization scheme not the Adler-Bardeen scheme.

dence. In fact, this behavior of L^{u-d} is related to the diverging behavior of the QCD running coupling constant $\alpha_S(Q^2)$ as $Q^2 \rightarrow 0$. If the magnitude of α_S becomes too large, one must suspect the validity of the used QCD evolution equation, which is based on the framework of *perturbative* renormalization group equation. Also very difficult to know is the size of ambiguity arising from the choice of the starting energy of evolution, because the renormalization scale of any effective model can be given only by a crude guess. Wakamatsu then advocated the following strategy. Instead of carrying out upward evolution by starting from the predictions of effective models corresponding to low energy scales, one may start with the information known at the high energy scales, say, at $Q^2 \simeq 4 \text{ GeV}^2$ and to carry out a downward evolution by leaving the question where to stop this downward evolution. The quantity L^{u-d} at the scale of $Q^2 = 4 \text{ GeV}^2$ can be estimated by using the relation

$$L^{u-d} = J^{u-d} - \frac{1}{2} \Delta \Sigma^{u-d}, \quad (218)$$

with

$$J^{u-d} = \frac{1}{2} [\langle x \rangle^{u-d} + B_{20}^{u-d}(0)]. \quad (219)$$

Here, $\Delta \Sigma^{u-d}$ is identified with the beta-decay coupling constant $g_A^{(I=1)}$ of the neutron, which is known with high precision. The difference $\langle x \rangle^{u-d}$ between the u - and d -quark momentum fractions at $Q^2 = 4 \text{ GeV}^2$ is also a fairly precisely known quantity from the global analysis of the inclusive DIS data.^{140,141} Only one unknown is therefore the isovector anomalous gravito-magnetic moment $B_{20}^{u-d}(0)$ of the nucleon. Fortunately, this is an isovector quantity, which receives no DI contribution, so that one can expect that the corresponding lattice QCD prediction by the LHPC or the QCDSF-UKQCD collaborations is much more reliable than that for $B_{20}^{u+d}(0)$. The value of L^{u-d} estimated in this way is used as a initial condition given at $Q^2 = 4 \text{ GeV}^2$, and the downward evolution was carried out to obtain the value of L^{u-d} at the low energy scales corresponding to effective models of the nucleon.

Fig.4 shows an example of such calculations.¹³⁸ One sees that the scale dependence of L^{u-d} is in fact quite strong as Q^2 becomes very low. At the unitarity violating limit ($Q^2 \simeq (0.225 \text{ GeV})^2$), $2 L^{u-d}$ becomes close to zero.^m However, at the favorite matching scale with the cloudy bag model, i.e. $Q^2 \simeq (0.4 \text{ GeV})^2$, one still finds that $2 L^{u-d}$ is sizably negative, i.e. $2 L^{u-d} \simeq -0.53$. This negative value with large magnitude is in sharp contradiction with the predictions of the refined cloudy bag model or any quark model with SU(6)-like spin-flavor structure. Very curiously, the prediction of the CQSM for $2 L^{u-d}$ given in the papers 142, 138 is negative with large magnitude, and it is consistent with the above phenomenological estimate with partial use of the lattice data. It would be interesting to point

^mThe unitarity violating limit here means the scale where the gluon momentum fraction becomes *negative*, when one performs a downward evolution by starting from the empirical values of quark and gluon momentum fractions given at $Q^2 = 4 \text{ GeV}^2$.

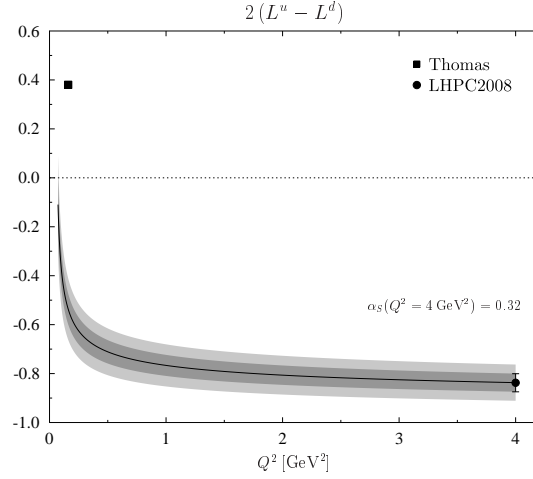


Fig. 4. The scale dependence of $2L^{u-d}$ obtained by solving the leading-order evolution equation with the LQCD prediction at $Q^2 = 4 \text{ GeV}^2$ with errors as initial condition of downward evolution. Also shown by the filled square is the prediction of the improved cloudy bag model corresponding to the scale $Q_0^2 = (0.4 \text{ GeV})^2$.

out the fact that the theoretical prediction for $2L^{u-d}$ given in these papers is not the canonical OAM. It is obtained through the calculation of the forward limit of $A_{20}^{u-d}(0)$ and $B_{20}^{u-d}(0)$ together with the Ji sum rule. It is therefore a quantity which is conceptually closer to the mechanical OAM rather than the canonical OAM, although the distinction between these two OAMs is not so clear for an effective quark model like the CQSM, which is not a gauge theory.

Does the strong scale dependence of L^{u-d} rescue the discrepancy between the prediction of the lattice QCD and that of the low energy models or not? Or does it suggest a significant numerical difference between the canonical and mechanical OAMs in the nucleon? The answer to this question is not yet absolutely clear. However, it seems clear that the above mentioned puzzle provides us with valuable nontrivial information on the role of quark OAMs in the nucleon spin decomposition problem.

9. Summary and concluding remarks

Now we are in a position to answer the proposed question in the present paper. “Is gauge-invariant complete decomposition of the nucleon spin possible?”ⁿ The truth appears that this question is a little bit too general to give a unique answer. If the question concerns the most general nucleon spin decompositions including the transverse spin sum rules, the answer is likely to be “No”. On the other hand, if the

ⁿThe gauge-invariance here should be taken as a traditional one, not as the weak gauge-invariance *a la* Lorc e.

question concerns the most fundamental longitudinal nucleon spin decomposition, the answer would most probably be “Yes”. The reason is the following. The two “seemingly” covariant gauge-invariant decompositions (I) and (II) of the nucleon spin proposed by Wakamatsu is of general nature in the sense that it still has a large degrees freedoms such that it can be reduced to any known gauge-invariant decompositions after an appropriate choice of the Lorentz frame of reference. In particular, there is no doubt about that the decomposition (II) contains the two popular gauge-invariant decompositions of Chen et al. and of Bashinsky-Jaffe, depending on an appropriate choice of Lorentz frame and a suitable condition which is necessary to uniquely specify the decomposition of the gauge field into the physical and pure-gauge components. Since each term of those two decompositions is separately gauge-invariant, both are clearly gauge-invariant decompositions. Remember the fact that, in the QED case, the Chen decomposition is nothing but the familiar transverse-longitudinal decomposition of the photon field and in particular that the transverse component is a gauge-invariant quantity with unambiguous physical meaning. Still, things to be worried about here is that the transverse-longitudinal decomposition or the concept of transversality of the gauge field is generally Lorentz-frame dependent concept. This is the reason of general statements found in many standard textbooks of electrodynamics, which tells that the total photon angular momentum cannot be gauge-invariantly decomposed into the orbital and intrinsic spin parts. This statement would certainly be true in the most general context. However, as shown in sect.5, this is not necessarily the case for the longitudinal component of the total photon angular momentum. The point is that the helicity for a massless particle is a Lorentz-invariant quantity.^o The component of the total photon angular momentum along the direction of the photon momentum can be decomposed into the orbital and intrinsic spin parts in a gauge- and frame-independent way and both are definite observables.

Coming back to our nucleon spin decomposition problem, the helicity sum rule of the nucleon is basically a Lorentz-frame independent sum rule. In particular, it is invariant under a wide class of Lorentz boost in the direction of the nucleon momentum^p. Because the longitudinal spin sum rule of the nucleon, or the helicity sum rule, is invariant under such Lorentz-boosts, one can work in any Lorentz-frame. This especially means that there is nothing wrong in working within a noncovariant framework like in the Chen decomposition. A logical conclusion drawn from this consideration is that the Chen decomposition and the Bashinsky-Jaffe decomposition (or the Hatta decomposition), which can be reduced from more general decomposition (II) *a la* Wakamatsu would give the same answer for the longitudinal decomposition of the nucleon spin. Note, however, that this is not true for

^oIn fact, there is an explicit proof that the photon helicity is an invariant, even though, in general, Lorentz boosts transform the transverse, longitudinal, and the time-like components of the vector potential into each other.¹⁴³

^pNaturally, since the nucleon is a massive particle, its helicity can change under an extremely fast Lorentz boost in the nucleon momentum direction.

more general nucleon spin decompositions like the transverse decomposition of the nucleon spin. At any rate, an important conclusion drawn from the consideration above is that the longitudinal gluon spin ΔG is most likely to be a gauge- and frame-independent observable. In other words, ΔG is a gauge-invariant quantity in a traditional or strong sense at variance with the statement in the review by Leader and Lorcé.⁸²

Another important subject addressed in the present review is the question of observability of the two kinds of OAMs of quarks and gluons, i.e. the mechanical OAM and the generalized (gauge-invariant) canonical OAM. Now it is a wide-spread belief in the QCD spin physics community that the canonical OAM (not the mechanical OAM) is the quantity with natural physical interpretation as OAMs of free partonic motion of constituents, i.e. quarks and gluons. There are two reasons for this belief. First, the generalized canonical OAMs are believed to obey the standard angular momentum commutation relation, so that they are supposed to work properly as generators of spatial rotation. Second, it is widely believed that the dynamical quark OAM is given as a sum of the canonical quark OAM and the potential angular momentum, which appears to support the interpretation that the dynamical quark OAM contains the genuine twist-3 quark-gluon interaction term. As explained in the present paper, both these beliefs are not necessarily justified. Concerning the first one, we have shown that, for a massless photon, neither of the canonical OAM nor the intrinsic spin satisfies the SU(2) algebra. It was also shown that this observation is inseparably connected with the fact that there is no rest frame for a massless particle. Concerning the second question, we have given a plausible argument to show that what contains the potential angular momentum is rather the canonical quark OAM than the dynamical quark OAM, in contradiction to naive expectation. These observations are by no means academic ones. In fact, they have important consequences on the possible observability of the two OAMs. Now we know that the canonical OAM is related to a certain moment of the Wigner distribution, which is expected to describe the partonic orbital motion of quarks in the plane perpendicular to the direction of nucleon momentum and spin. However, the recent paper by Courtoy et al. revealed a principle difficulty of observing the relevant distribution appearing in this sum rule.¹⁰⁵ On the other hand, the dynamical OAMs are already known to have clear relations to DIS observables. One is the indirect relations through the GPDs and the Ji sum rule. The other is the relation in which the dynamical quark OAM is given as a 2nd moment of the GPD G_2 . Although the actual experimental determination of G_2 would not be an easy task, an interesting fact is that the genuine twist-3 part of G_2 does not contribute to this sum rule and that the Wandzura-Wilczek part of G_2 is completely determined by the twist-2 GPDs $H^q(x, 0, 0)$, $E^q(x, 0, 0)$, and $\tilde{H}^q(x, 0, 0)$. This appears to support our viewpoint that what contains the genuine twist-3 quark-gluon interaction term is not the dynamical quark OAM but the canonical quark OAM.

To sum up, what discriminates the two gauge-invariant decompositions of the nucleon spin are the orbital angular momentum parts of quarks and gluons. They

are specified by the two different OAMs, i.e. the dynamical OAM and the canonical OAM. For a weakly-coupled gauge system like the hydrogen atom, there is no practical difference between these two OAMs and there is nothing wrong in believing that the canonical OAM is a natural building block of quantum theory. This is because the weak interactions between the transverse photons and the charged particles can be introduced and handled at later stage as a perturbation. For a strongly-coupled gauge systems like the nucleon, however, the distinction between the canonical OAM and the dynamical OAM becomes crucial. Here, we cannot neglect the Fock-components of the transverse gluon in the nucleon wave function. Otherwise, we would have no gluon distributions. This means that, when one talks about the OAMs of quarks and gluons in the nucleon, one must at the least be clearly conscious of which OAMs one is thinking of.

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References

1. EMC Collaboration : J. Aschman et al., *Phys. Lett. B* **206**, 364 (1988).
2. EMC Collaboration : J. Aschman et al., *Nucl. Phys. B* **328**, 1 (1989).
3. M. Anselmino, A. Efremov, and E. Leader, *Phys. Rep.* **261**, 1 (1995).
4. H.-Y. Cheng, *Int. J. Mod. Phys. A* **11**, 5109 (1996).
5. B. Lampe and E. Reya, *Phys. Rep.* **332**, 1 (2000).
6. B.W. Filipponi and. X. Ji, *Adv. Nucl. Phys.* **26**, 1 (2001).
7. S. Bass, *Rev. Mod. Phys.* **77**, 1257 (2005).
8. S.E. Kuhn, J.-P. Chen, and E. Leader, *Prog. Part. Nucl. Phys.* **63**, 1 (2009).
9. M. Burkardt, C.A. Miller, and W.D. Nowak, *Rep. Prog. Phys.* **73**, 016201 (2010).
10. C.A. Aidala, S.D. Bass, D. Hasch, and G.K. Mallot, *Rev. Mod. Phys.* **85**, 655 (2013).
11. COMPASS Collaboration : E. S. Ageev et al., *Phys. Lett. B* **633**, 25 (2006).
12. PHENIX Collaboration : K. Boyle et al, *AIP Conf. Proc.* **842**, 351 (2006).
13. STAR Collaboration : J. Kirelyuk et al., *AIP Conf. Proc.* **842**, 327 (2006).
14. STAR Collaboration : R. Fatemi et al., nucl-ex/0606007 (2006).
15. G. Altarelli and G.G. Ross, *Phys. Lett. B* **212**, 391 (1988).
16. R.D. Carlitz, J.C. Collins, and A.H. Mueller, *Phys. Lett. B* **214**, 229 (1988).
17. A.V. Efremov and O.V. Teryaev, *JINR Dubna preprint*, JINR EZ-88-297 (1988).
18. COMPASS Collaboration : E. S. Ageev et al., *Phys. Lett. B* **612**, 154 (2005).
19. COMPASS Collaboration : V. Yu. Alexakhin et al., *Phys. Lett. B* **647**, 8 (2007).
20. HERMES Collaboration : A. Airapetian et al., *Phys. Rev. D* **75**, 012007 (2007).
21. R.L. Jaffe and A. Manohar, *Nucl. Phys. B* **337**, 509 (1990).
22. X. Ji, *Phys. Rev. Lett.* **78**, 610 (1997).
23. X. Ji, *J. Phys. G* **24**, 1181 (1998).
24. O. V. Teryaev, hep-ph/9803403 (1998).
25. S.V. Bashinsky and R.L. Jaffe, *Nucl. Phys. B* **536**, 303 (1999).
26. G.M. Shore and B.E. White, *Nucl. Phys. B* **581**, 409 (2000).

27. B.L.G. Bakker, E. Leader, and T.L. Trueman, *Phys. Rev. D* **70**, 114001 (2004).
28. X. S. Chen, X. F. Lü, W. M. Sun, F. Wang, and T. Goldman, *Phys. Rev. Lett.* **100**, 232002 (2008).
29. X. S. Chen, W. M. Sun, X. F. Lü, F. Wang, and T. Goldman, *Phys. Rev. Lett.* **103**, 062001 (2009).
30. C.W. Wong, Fan Wang, W.M. Sun, and X.F. Lü, arXiv : 1010.4336 [hep-ph] (2010).
31. Fan Wang, X. S. Chen, X. F. Lu, W. .M. Sun, and T. Goldman, *Nucl. Phys.* **A844**, 85c (2010).
32. W. M. Sun, X. S. Chen, X. F. Lu, F. Wang, *Phys. Rev. A* **82**, 012107 (2010).
33. X. S. Chen, W. M. Sun, F. Wang, and T. Goldman, *Phys. Rev. D* **83**, 071901 (2011).
34. X. S. Chen, W. M. Sun, F. Wang, and T. Goldman, *Phys. Lett.* **B700**, 21 (2011).
35. T. Goldman, *AIP Conf. Proc.* **1418**, pp. 13-20 (2011).
36. X. S. Chen, arXiv : 1203.1288 [hep-ph] (2012).
37. S. C. Tiwari, arXiv : 0807.0699 [hep-ph] (2008).
38. S. C. Tiwari, *Phys. Rev. D* **88**, 065022 (2013).
39. X. Ji, *Phys. Rev. Lett.* **104**, 039101 (2010).
40. X. Ji, *Phys. Rev. Lett.* **106**, 259101 (2011).
41. M. Wakamatsu, *Phys. Rev. D* **81**, 114010 (2010).
42. M. Wakamatsu, *Phys. Rev. D* **83**, 014012 (2011).
43. M. Wakamatsu, *Phys. Rev. D* **84**, 037501 (2011).
44. M. Wakamatsu, *Journal of Physics: Conference Series* **296**, 012038 (2011).
45. M. Wakamatsu, *Phys. Rev. D* **85**, 114039 (2012).
46. M. Wakamatsu, *Phys. Rev. D* **87**, 094035 (2013).
47. E. Leader, *Phys. Rev. D* **83**, 096012 (2011) [*Erratum-ibid. D* **85**, 039905 (2012)].
48. E. Leader, *Phys. Rev. D* **85**, 051501(R) (2012).
49. E. Leader and C. Lorcé, *Phys. Rev. Lett.* **111**, 039101 (2013).
50. E. Leader, *Phys. Lett. B* **720**, 120 (2013).
51. C. Lorcé, *Phys. Rev. D* **87**, 034031 (2013).
52. C. Lorcé, *Phys. Lett. B* **719**, 185 (2013).
53. C. Lorcé, *Phys. Rev. D* **88**, 044037 (2013).
54. C. Lorcé, arXiv 1306.0456 [hep-ph] (2013).
55. C. Lorcé, arXiv 1401.3954 [hep-ph] (2014).
56. Y. Hatta, *Phys. Rev. D* **84**, 041701(R) (2011).
57. Y. Hatta, *Phys. Lett. B* **708**, 186 (2012).
58. Y. Hatta and S. Yoshida, *JHEP* **10**, 080 (2012).
59. Y. Hatta, K. Tanaka, and S. Yoshida, *JHEP* **02**, 003 (2013).
60. M. Burkardt and H. BC, *Phys. Rev. D* **79**, 071501 (2009).
61. M. Burkardt, *Phys. Rev. D* **72**, 094020 (2005).
62. M. Burkardt, *Phys. Rev. D* **88**, 014014 (2013).
63. Y. M. Cho, Mo-Lin Ge, D. G. Pak, and Pengming Zhang, arXiv : 1102.1130 [nucl-th] (2011).
64. Y. M. Cho, *AIP Conf. Proc.* **1418**, pp. 29-38 (2011).
65. Y.M. Cho, Mo-Lin Ge, and Pengming Zhang, *Mod. Phys. Lett. A* **27**, 1230032 (2012).
66. P. M. Zhang and D. G. Pak, *Eur. Phys. J. A* **48**, 91 (2012).
67. D.G. Pak and P.M. Zhang, arXiv : 1204.5030 [hep-ph] (2012).
68. Z.-Q. Guo and I. Schmidt, *Phys. Rev. D* **87**, 114016 (2013).
69. Z.-Q. Guo and I. Schmidt, *Phys. Rev. D* **87**, 114017 (2013).
70. B.-H. Zhou and Y.-C. Huang, *Phys. Rev. D* **84**, 047701 (2011).
71. B.-H. Zhou, C. Huang, and Y.-C. Huang, unpublished (2012).
72. H.-W. Lin and K.-F. Liu, *Phys. Rev. D* **85**, 058901 (2012).

73. X. Ji, Y. Xu, and Y. Zhao, *JHEP* **1208**, 082 (2012).
74. X. Ji, X. Xiong, and F. Yuan, *Phys. Rev. Lett.* **109**, 152005 (2012).
75. X. Ji, X. Xiong, and F. Yuan, *Phys. Lett. B* **717**, 214 (2012).
76. X. Ji, J.-H. Zhang, and Y. Zhang, *Phys. Rev. Lett.* **111**, 112002 (2013)
77. X. Ji, *Phys. Rev. Lett.* **110** 262002 (2013).
78. Y. Hatta, X. Ji, and Y. Zhao, arXiv : 1310.4263 [hep-ph] (2013).
79. X. Xiong, X. Ji, J.H. Zhang, and Y. Zhao, arXiv : 1310.7471 [hep-ph] (2013).
80. A. Harindranath, R. Kundu, and A. Mukherjee, and R. Rotabole, *Phys. Rev. Lett.* **111**, 039102 (2013).
81. A. Harindranath, R. Kundu, and A. Mukherjee, *Phys. Lett. B* **728**, 63 (2014).
82. E. Leader and C. Lorcé, arXiv 1309.4235 [hep-ph] (2013).
83. R.P. Feynman, R.B. Leighton, and M.L. Sands, *Feynman Lectures on Physics, Vol.III* (Addison-Wesley, Reading, MA, 1965).
84. P. Hoodbhoy and X. Ji, *Phys. Rev. D* **60**, 114042 (1999)
85. E.C.G. Stueckelberg, *Helv. Phys. Acta.* **11**, 299 (1938).
86. S. Kamefuchi and H. Umezawa, *Nuovo Cimento* **32**, 448 (1964).
87. T. Kunimasa and T. Goto, *Prog. Theor. Phys.* **37**, 452 (1967).
88. J.D. Bjorken and S.D. Drell, *Relativistic Quantum Fields*, (MacGraw-Hill, Inc., New York, 1965).
89. J.K. Lubanski, *Physica (Amsterdam)* **9**, 310 (1942).
90. W.N. Polyzou, W. Glöckle, and H. Witala, *Few Body Syst.* **54**, 1667 (2013).
91. S. J. Van Enk and G. Nienhuis, *J. Mod. Opt.* **41**, 963 (1994).
92. S. J. Van Enk and G. Nienhuis, *Europhys. Lett.* **25**, 497 (1994).
93. C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms* (John Wiley & Sons Inc. 1989).
94. J.M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Springer-Verlar, Berlin 1976).
95. V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, *Quantum Electrodynamics* (Pergamon, Oxford 1982).
96. R.A. Beth, *Phys. Rev.* **50**, 115 (1936).
97. L. Allen, M.W. Beijersbergen, R.J.C. Spreeuw, and J.P. Woerdman, *Phys. Rev. A* **45**, 8185 (1992).
98. S.J. Van Enk and G. Nienhuis, *Optics Commun.* **94**, 146 (1992).
99. E. Leader, *Spin in Particle Physics* (Cambridge University Press, Cambridge 2001).
100. J.J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley Pub. Co. 1995) Chap. 2.6.
101. E.J. Konopinski, *Am. J. Phys.* **46(5)**, 499 (1978).
102. C. Lorcé and B. Pasquini, *Phys. Rev. D* **84**, 014015 (2011).
103. S. Meissner, A. Metz, and M. Schlegel, *JHEP* 0908, 056 (2009).
104. C. Lorcé and B. Pasquini, *JHEP* **1309**, 138 (2013).
105. A. Courtoy, G.R. Goldstein, J.O. Gonzalez Hernandez, S. Liuti, and A. Rajan, arXiv : 1310.5157 [hep-ph] (2013).
106. D. Sivers, *Phys. Rev. D* **41**, 83 (1990).
107. D. Sivers, *Phys. Rev. D* **43**, 261 (1991).
108. M. Penttinen, M.V. Polyakov, A.G. Shuvaev, and M. Strikman, *Phys. Lett. B* **491**, 96 (2000).
109. D.V. Kiptily and M.V. Polyakov, *Eur. Phys. J. C* **37**, 105 (2004).
110. A.V. Belitsky, D. Mueller, and A. Kirchner, *Nucl. Phys. B* **629**, 323 (2002)
111. D. Boer, P.J. Mulders, and F. Pijlman, *Nucl. Phys. B*, 201 (2003).
112. J. Qiu and G. Sterman, *Phys. Rev. Lett.* **67**, 2264 (1991).

113. J.C. Collins, *Phys. Lett. B* **536**, 43 (2002).
114. R. L. Jaffe, *Phil. Trans. R. Soc. Lond. A* **359**, 391 (2001).
115. LHPC Collaboration : Ph. Hägler et al., *Phys. Rev. D* **77**, 094502 (2008).
116. LHPC Collaboration : J.D. Bratt et al., *Phys. Rev. D* **82**, 0940502 (2010).
117. Ph. Hägler, *Phys. Rep.* **490**, 49 (2010).
118. A.Ali Khan et al., hep-lat/0603028.
119. Dirk Brömmel et al., *Pos. LAT2007* p.158.0710.1534 (2007).
120. M. Wakamatsu and Y. Nakakoji, *Phys. Rev. D* **74**, 054006 (2006).
121. M. Wakamatsu and Y. Nakakoji, *Phys. Rev. D* **77**, 074011 (2008).
122. H. Weigel, L. Gamberg, and H. Reinhardt, *Mod. Phys. Lett. A* **11**, 3021 (1996).
123. H. Weigel, L. Gamberg, and H. Reinhardt, *Phys. Lett. B* **399**, 286, (1997).
124. L. Gamberg, H. Reinhardt, and H. Weigel, *Phys. Rev. D* **58**, 0540014 (1998)
125. D.I. Diakonov, V.Yu. Petrov, P.V. Pobylitsa, M.V. Polyakov, and C. Weiss, *Nucl. Phys. B* **480**, 341 (1996).
126. D.I. Diakonov, V.Yu. Petrov, P.V. Pobylitsa, M.V. Polyakov, and C. Weiss, *Phys. Rev. D* **56**, 4069 (1997).
127. P.V. Pobylitsa, M.V. Polyakov, K. Goeke, T. Watabe, and C. Weiss, *Phys. Rev. D* **59**, 034024 (1999).
128. M. Wakamatsu and T. Kubota, *Phys. Rev D* **57**, 5755 (1998).
129. M. Wakamatsu and T. Kubota, *Phys. Rev. D* **60**, 034020 (1999).
130. M. Wakamatsu, *Phys. Rev. D* **67**, 034005 (2003).
131. M. Wakamatsu, *Phys. Rev. D* **67**, 034006 (2003).
132. M. Wakamatsu and H. Yoshiki, *Nucl. Phys. A* **524**, 561 (1991).
133. M. Wakamatsu and T. Watabe, *Phys. Rev. D* **62**, 054009 (2000).
134. χ QCD Collaboration : K.F. Liu et al., Pos LATTICE **2012**, 164 (2011).
135. χ QCD Collaboration : M. Deka et al., arXiv : 1312.4816 [hep-lat] (2013).
136. A.W. Thomas, *Phys. Rev. Lett.* **101**, 102003 (2008).
137. F. Myhrer and A.W. Thomas, *Phys. Rev. D* **38**, 1633 (1988).
138. M. Wakamatsu, *Eur. Phys. J. A* **44**, 297 (2010).
139. X. Ji, J. Tang, and P. Hoodbhoy, *Phys. Rev. Lett.* **76**, 740 (1996).
140. A.D. Martin, W.J. Stirling, and R.S. Thorne, *Phys. Lett.* **636**, 259 (2006).
141. L.H. Lai, J. Huston, S. Kuhlmann, J. Martin, F. Olness, J.F. Owen, J. Pumplin, and W.K. Tung, *Eur. Phys. J. C* **12**, 375 (2000).
142. M. Wakamatsu and H. Tsujimoto, *Phys. Rev. D* **71**, 074001 (2005).
143. K. Haller and R.B. Sohn, *J. Math. Phys.* **19**, 1589 (1978)